RESEARCH STATEMENT

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I work at the intersection of harmonic analysis, geometry, and dynamics, and I study two primary objects: translation surfaces and metric graphs. I study the relationship between the moduli space of translation surfaces and individual translation surfaces, as well as spectral theory on translation surfaces and metric graphs. The majority of my work involves connecting harmonic analysis to *dynamical* and *algebraic* properties of these objects. On both types of objects the Laplacian turns out to be difficult to study. The existence of singularities, despite how mild, creates difficulties in the spectral and representation theory. Often, the strategy is to look elsewhere for the right application of harmonic analysis. As such, much of my work involves the harmonic analysis and representation theory of $SL_2(\mathbb{R})$, including bounding matrix coefficients with the Harish-Chandra function, employing the Kunze-Stein phenomenon to understand decay of $SL_2(\mathbb{R})$-averages, studying the characters (or spherical functions) of the C*-algebra of bi-K-invariant measures on $SL_2(\mathbb{R})$, and more.

Definitions and outline. A *translation surface* is a polygon or a set of polygons in the plane such that each side of the polygon(s) is identified to a parallel side by translation [37], [38]. Equivalently, the data of a translation surface is an ordered pair (X, ω), where X is a compact, connected Riemann surface and ω a non-zero holomorphic differential. Translation surfaces are flat away from a finite set of singular points. The singular points are cone points whose angles are integer multiples of 2π. Translation surfaces come with a natural action of $SL_2(\mathbb{R})$, where the action of a matrix is just the usual linear action. Since the linear action sends parallel lines to parallel lines, the action sends translation surfaces to translation surfaces. We call the stabilizer of this action the *Veech group* of the surface. For an example of a stabilizing element, consider the unit square with opposite sides identified by translation (a torus) and let $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

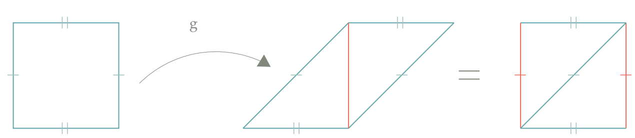

FIGURE 1. Stabilizing element of the Veech group

Using the cut-and-paste procedure pictured in Figure 1, we can reassemble the new polygon as the old, meaning the underlying topological space is the same. This example shows us that the Veech group is not always trivial. Visually, the action of the matrix appears related to linear maps on the surface, and in fact, this is true. We can identify the Veech group with the collection of derivatives of affine self-diffeomorphisms of the surface [36], up to a finite set of automorphisms of the translation structure. (For experts, I am thinking of the torus with a marked point.)

In this setting, I have been studying the diophantine properties of affine diffeomorphisms of translation surfaces using a shrinking target framework (see 1.1). The work on groups of affine diffeomorphisms also led to an effective weak-mixing result (see 1.2). These results are deduced by using an affine invariant manifold (an $SL_2(\mathbb{R})$-orbit closure) in the *moduli space of translation surfaces*. (For experts, we work in the moduli space of marked surfaces.) This is a theme in the field: dynamical and geometric properties of the moduli space can control dynamical properties on individual translation surfaces. The results are interesting because it is the first time (that we are aware of) that quantitative properties of the group of affine diffeomorphisms are deduced using an argument that crucially uses an affine invariant manifold the moduli space. Much of this is joint work with Chris Judge [18] [19].

The relationship between the moduli space and individual surfaces has been fruitful in other ways. Consider, for example, Masur's Criterion which tells us that if we pick a translation surface in the moduli space, flow by the geodesic flow, and observe that the trajectory returns infinitely often to a compact set in the moduli space, then we are guaranteed to have a uniquely ergodic vertical (north) linear flow on the translation surface. Along these lines, I have shown that there is a quantitative density condition on the linear flow of a particular set of translation surfaces, *superdensity*, that is both necessary and sufficient to know that the associated geodesic trajectory in the moduli space never leaves a compact set [33]. This is discussed in subsection 1.3.

Whereas superdense trajectories on a translation surface satisfy a quantitative density condition, at the other extreme we have closed trajectories of the linear flow. Closed trajectories of the linear flow are related to cylinders of a translation surface (closed trajectories foliate cylinders). Finding cylinder decompositions on a generic translation surface is, in general, hard. In subsection 1.4, we take a detour to a study of infinite translation surfaces, and show that for a particular type of finite area surface, you can inductively construct a (non-obvious) cylinder decomposition of the surface using observations about the geometry of the surface. This is joint work with Dami Lee [24].

Bounding the sides of a cylinder are saddle connections, linear segments whose endpoints are cone points on the surface. Athreya and Cheung initiated the study of the slope gaps between saddle connections by recognizing that the work of Boca, Cobeli, and Zaharescu on Farey fractions can be recast in a dynamical setting, and that the gaps between Farey fractions are the slope gaps of the square-torus. When appropriately normalized, these slope gaps converge to a particular distribution [2] [21]. In subsection 1.5, we effectivize this convergence. This work has led to many questions. How can we better count lattice points in expanding triangles in the plane? Can we prove the prime number theorem with our effective result? And the much harder question, closer to the heart of ongoing research on translation surfaces: how do long closed horocycles equidistribute in an affine invariant manifold. This is joint work with Tariq Osman and Jane Wang [27].

A *metric graph* is a compact, connected metric space such that any point has a neighborhood isometric to a star shaped set. In other words, it is a one-dimensional manifold, except at finitely many points which are isomorphic to stars with more than two branches.

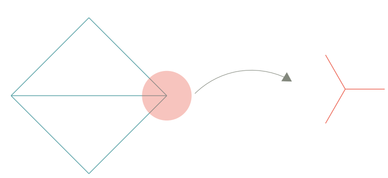

FIGURE 2. Chart to a star-shaped set

I am involved in ongoing work with Junaid Hasan and Farbod Shokrieh in which we study methods of computing analytic torsion on metric graphs with respect to a cohomology that arises from the theory of Chambert-Loir and Ducros [8]. This study has connections to tropical geometry, and touches on a deeper question concerning mysterious analogies between tropical and complex geometry. This work is discussed in Section 2.

1. Translation Surfaces

1.1. **Diophantine properties of affine diffeomorphisms.** In 1989, Veech discovered a class of translation surfaces that have "large" Veech groups, specifically, Veech groups that are lattices in $SL_2(\mathbb{R})$ [36]. Such lattices are necessarily non-cocompact, finite covolume discrete subgroups of $SL_2(\mathbb{R})$ [17]. We call these surfaces *lattice surfaces*. Veech groups of lattice surfaces contain a hyperbolic element, which can be represented as a matrix with expanding and contracting eigenspaces. The affine diffeomorphism corresponding to this element, after several applications, sufficiently "mixes" the points on the surface. In fact, the map will be *weakly mixing* (with respect to the Lebesgue measure on the surface). For example, consider Arnold's cat map shown in Figure 3. One can imagine that after several iterations, the cat in Figure 3 will be quite blurred, illustrating that the points are moving around substantially.

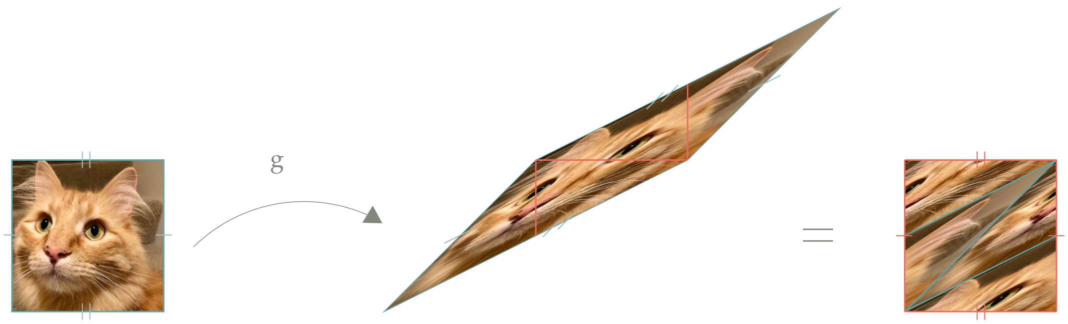

FIGURE 3. Arnold's Cat Map

Since the Veech group of any lattice surface (X, ω) contains a hyperbolic element, the action of the group of affine diffeomorphims, $\text{Aff}_\omega(X)$, is *ergodic* (and in fact, *weakly mixing*). Hence, we can ask questions about the *density* of the orbits of this action. One way to answer this is by proving that the action $\text{Aff}_\omega(X)$ exhibits a *shrinking target property*. Fix a lattice surface (X, ω) with affine diffeomorphisms $\text{Aff}_\omega(X)$, and pick any $y \in X$. Let $B_g(y)$ denote the open ball of radius $\phi(||g||)$ (a decreasing function of the operator norm). Does almost every $x \in X$ have the property that $g \cdot x \in B_g(y)$ for infinitely many $g \in \text{Aff}_\omega(X)$? How fast can ϕ decrease (the target shrink) before this no longer holds?

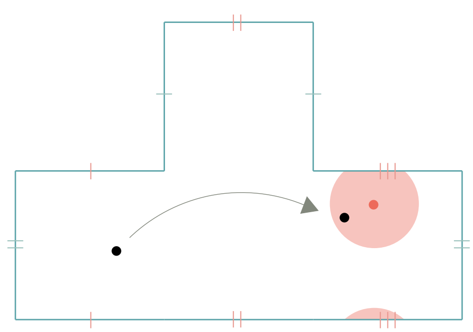

FIGURE 4. Hitting the target

Consider the linear action of $SL_2(\mathbb{Z})$ on the unit torus. The torus is an example of a translation surface and $SL_2(\mathbb{Z})$ is its Veech group. In fact, $SL_2(\mathbb{Z})$ is a lattice subgroup of $SL_2(\mathbb{R})$, so the torus is an example of a lattice surface. Moreover, the group of affine diffeomorphisms which fix a point on the surface is $SL_2(\mathbb{Z})$. Our interest is in connecting the dynamics of the affine diffeomorphisms to the Laplacian on the torus. The action of $SL_2(\mathbb{Z})$ on the surface induces a group representation, the *Koopman representation*, $\pi : SL_2(\mathbb{Z}) \to \mathcal{B}(L^2(\mathbb{T}^2))$, where $\pi(g)f(x) = f(g^{-1}x)$. Recall that the eigenfunctions of the Laplacian, $\Delta = -(\partial_x^2 + \partial_y^2)$, are solutions to $\Delta f = \lambda f$. We can compute eigenfunctions: $e^{2\pi imx}e^{2\pi iny}$, where $(m,n) \in \mathbb{Z}^2$. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, then

$$\pi(g)e^{2\pi imx}e^{2\pi iny} = e^{2\pi i(dm-cn)x}e^{2\pi i(an-bm)y}.$$

This is significant: the Koopman representation sends eigenspaces of the Laplacian to eigenspaces. In other words, *the action of the group of Affine diffeomorphisms plays nicely with the spectral properties of the Laplacian.*

In 2016, Finkelshtein used this property coupled with a spectral gap result that he proved to show that the affine diffeomorphisms of the torus exhibit diophantine-like behavior [13]. In my thesis, I lifted this result to regular square-tiled surfaces. A *square-tiled surface* is a branched cover over the torus, branched only at one point. Note that a square-tiled surface is so named because we can tile the surface by squares. See Figure 4. A *regular* square-tiled surface is a square-tiled surface whose automorphisms of the translation structure act transitively on the squares.

Theorem 1.1 (S, 24)**.** Let (X, ω) be a regular square-tiled surface, and let Γ be a subgroup of the subgroup of $\mathrm{Aff}_\omega(X)$ that fixes all of the cone points. For any $y \in X$, for Lebesgue a.e. $x \in X$, the set

$$\{\phi \in \Gamma : |\phi(x) - y| < ||D\phi||^{-\alpha}\}$$

is

(1) finite for every $\alpha > \delta_\Gamma$
(2) infinite for every $\alpha < \delta_\Gamma$

where δ_Γ is the critical exponent of the subgroup Γ, and $||\cdot||$ is the operator norm.

To prove this, we use a shrinking target framework. It is well known that there is a finite index subgroup of $SL_2(\mathbb{Z})$ which acts on the square-tiled surface [16], so we reduce consideration to this subgroup. Then, if our shrinking target on the square-tiled surface is the preimage of a shrinking target on the torus under the covering map, and the target on the torus is hit infinitely often by this subgroup, then the target on the square-tiled surface is hit infinitely often as well. That means that part of the target in *one of the squares* must be hit infinitely often. In other words, the property can be lifted provided the target on the square-tiled surface is evenly covering a target on the torus.

To remove the constraint of needing an evenly covered target, we use the automorphisms which act transitively on the squares and deduce that if one square is hit infinitely often, then every square is hit infinitely often. More generally, if we do not have a transitive automorphism group, we can form equivalence classes of squares (where two squares are equivalent if there exists an automorphism mapping one to the other), and deduce that *at least one* of the equivalence classes must be hit infinitely often.

Question 1.1. Is this just an artifact of the proof? Can we extend the rate to all squares? If not, for specific surfaces, can we identify which equivalence class inherits the hitting rate from the torus?

There is another question that follows from this work:

Question 1.2. can one extend the result to a larger class of surfaces?

Chris Judge and I have given a partial answer to this. We encode the data of the action of $\mathrm{Aff}_\omega(X)$ on the translation surface in an induced $SL_2(\mathbb{R})$-action on a bundle. The action of $SL_2(\mathbb{R})$ on the fibers of the bundle reproduces the action of $\mathrm{Aff}_\omega(X)$ on the translation surface X. For an

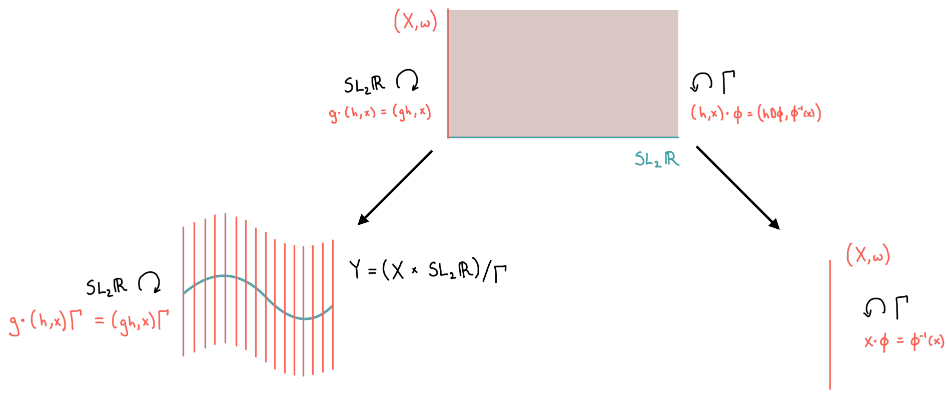

FIGURE 5. Induced bundle

example of such a bundle, let $\Gamma \subset \text{Aff}_\omega(X)$ be the subgroup fixing the cone points, and see Figure 5.

It just so happens that the bundle is an affine invariant manifold for a translation surface with a marked point. By studying the action in the bundle, we trade the difficulties of the harmonic analysis on an individual translation surface for the representation theory of $SL_2(\mathbb{R})$.

Coupled with Avila and Gouëzel's remarkable work [3], earlier work of Eskin-Masur [12] and Athreya's thesis work [1], we can use the bundle to prove the following.

Theorem 1.2. Assume (X, ω) is a lattice surface and let Y be the induced bundle with spectral parameter s, where $(1 - s^2)/4 = \lambda$ is the bottom of the spectrum of the Casimir operator on Y. Fix any point $y \in X$. The for almost every $x \in X$, the set

$$\left\{\phi \in \text{Aff}_\omega : \phi(x) - y < \|D\phi\|^{-\alpha}\right\}$$

is

(1) finite when $\alpha > 1$,
(2) infinite when $\alpha < 1 - s$.

Observe that there is a gap in the non-tempered case (when the bottom of the spectrum is below $\frac{1}{4}$). This gap is elucidating the weakness in our approach. The action of $SL_2(\mathbb{R})$ on the bundle induces a representation of $SL_2(\mathbb{R})$ on the set of unitary operators of $L^2(Y)$. The representation decomposes into irreducibles. Moreover, the Casimir operator (see [3]) commutes with the action of $SL_2(\mathbb{R})$, which is just a fancy way of saying that the irreducible components of the representation correspond to the eigenspaces of this operator. (By quotienting by $SO(2)$, the Casimir operator will descend to the hyperbolic Laplacian on $\mathbb{H} \cong SL_2(\mathbb{R})/\Gamma$.) This means that iterates of the $SL_2(\mathbb{R})$-action necessarily fall prey to the smallest eigenvalue, or rather, the bottom of the spectrum of the operator. However, the spectrum of this operator is not necessarily present before inducing the bundle, so there is no reason to think that the action of $\text{Aff}_\omega(X)$ is truly constrained by that spectrum. What we are seeing in the theorem is that the eigenvalues less than $\frac{1}{4}$ start creating a gap between the finite and infinite cases.

Question 1.3. Is there a more direct proof which explicitly uses the Γ-action on the translation surface? Alternatively, could one use a random walk approach on the affine-invariant manifold?

There is reason to believe that one of these methods is possible, but, in the first case, one would likely have to engage more directly with the harmonic analysis of the translation surface. An alternative to this may be engaging directly with the cocycle in the bundle without appealing to the representation theory of $SL_2(\mathbb{R})$. This may be possible by looking at random walks of $SL_2(\mathbb{R})$.

There is another lingering question.

Question 1.4. Can we extend the result to include arbitrary subgroups of the group of affine diffeomorphisms similar to how the result for square-tiled surfaces works for arbitrary subgroups?

One way to do this would be to use a very general, effective mean ergodic theorem for the action of $SL_2(\mathbb{R})$ on the bundle (see, for instance, [15]) to deduce a mean ergodic theorem for the action of radial sets intersected with the convex core of a convex cocompact subgroup by using the Kunze-Stein phenomenon (see [15]). Convex cocompact subgroups are "dense" inside of $SL_2(\mathbb{R})$, so that would allow one to bootstrap a result for arbitrary subgroups. However, what would be more interesting would be proving a mean ergodic theorem that has been tailored to this specific case, without relying on the Kunze-Stein phenomenon.

Question 1.5. Can one extend the result beyond lattice surfaces? To Arnoux-Yoccoz surfaces, or other such surfaces with large groups of affine diffeomorphisms?

To do this, one would need to work with affine invariant manifolds which are not necessarily bundles over a finite volume hyperbolic surface. This moves us closer to the state of the art in the field of translation surfaces, which involves the study of affine invariant manifolds, objects which are not homogeneous. Alternatively, one would need to directly engage with the harmonic analysis on a translation surface.

1.2. **Effective weak-mixing for affine diffeomorphisms.** Using a methodology of the induced bundle described above, a mean ergodic theorem holds for the action of $SL_2(\mathbb{R})$ on the affine invariant manifold associated to a twice-marked lattice surface. Again, the remarkable results of Avila and Gouëzel hold in this setting [3]. We are able to use the bundle construction above to compute the decay of averages of $\mathrm{Aff}_\omega(X)$ on product of lattice surfaces, less the diagonal, $X \times X \setminus \Delta$. This leads to the following result. Let $\Gamma \subset \mathrm{Aff}_\omega(X)$ be the subgroup that fixes the cone points. Let $G_t \subset SL_2(\mathbb{R})$ be bi-K-invariant sets whose measure grows exponentially in t, where the measure is the Haar measure on $SL_2(\mathbb{R})$. Let $\Gamma_t = \Gamma \cap G_t$. Let μ be the Lebesgue measure on the lattice surface (X, ω).

Theorem 1.3 (Weak-mixing of Γ-action)**.** For any $f, h \in L^2(X, \mu)$, there exists a $C > 0$ such that we have

$$\frac{1}{|\Gamma_t|} \sum_{\gamma \in \Gamma_t} \left| \int_X f(\gamma^{-1}x)h(x)\, d\mu - \int_X f\, d\mu \int_X h\, d\mu \right| \leq C\|f,h\|^{\frac{1}{2}} e^{-\frac{1}{2}\xi t}$$

where $\|f,h\| = \|h\|_2^2 + \|h\|_2 \cdot \|f \otimes h\|_1$ and $\xi = (5\sqrt{2} - 7)\theta$, and θ is the rate of decay for the mean ergodic theorem for $SL_2(\mathbb{R})$ acting on $SL_2(\mathbb{R})/\Gamma$.

1.3. **Linear flow on a translation surface.** One reason to be interested in affine diffeomorphisms is the relationship certain affine diffeomorphisms have with the linear flow on a translation surface, another commonly studied dynamical system. This is the usual geodesic flow on the translation surface, where, if a trajectory hits a cone point, we stop.

If we study the linear flow in a direction that corresponds to an eigendirection of a pseudo-Anosov affine diffeomorphism (one whose derivative is a hyperbolic element in $SL_2(\mathbb{R})$), then by applying the map, we can shrink or extend linear segments in those directions. In other words, renormalization dynamics, by way of these affine maps, is a tool we can use to study the linear flow in certain directions.

However, *the moduli space of translation surfaces* allows us to use renormalization to study the linear flow in *any* direction. In this section, I will highlight a new relationship between the linear flow on a translation surface and the renormalization dynamics in the moduli space.

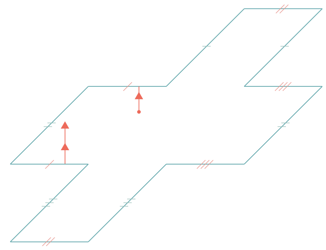

FIGURE 6. Linear flow segment on a translation surface

Recall that a translation surface is a pair (X, ω) where X is a compact, connected Riemann surface without boundary and ω a non-zero holomorphic differential on X. If we fix the genus of the underlying Riemann surface, *the moduli space* Ω_g of pairs (X, ω) forms a vector bundle over $\mathcal{M}_g$, the moduli space of genus g Riemann surfaces, where the fiber over $X \in \mathcal{M}_g$ is the g-complex dimensional vector space $\Omega(X)$ of holomorphic 1-forms on X. We will suppress the notation of the underlying Riemann surface and use the notation ω to denote a translation surface for the remainder of this subsection.

The $SL_2(\mathbb{R})$ action defined in the introduction is an action on the moduli space of translation surfaces. Notice that elements of the form $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ for any $t \in \mathbb{R}$ form a one-parameter subgroup. We will refer to this as the (Teichmüller) *geodesic flow*, and we will use this flow to renormalize linear trajectories on individual translation surfaces.

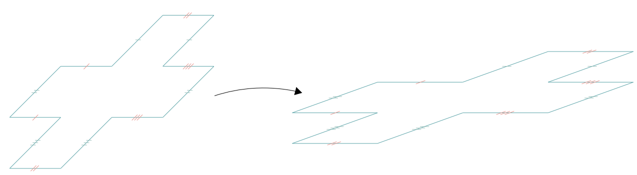

FIGURE 7. Translation surface ω and $g_t\omega$

We identify a quantitative density condition on the vertical flow of a translation surface ω that implies the diameter of $g_t\omega$ remains bounded for all $t > 0$. Moreover, this condition implies that the associated Teichmüller geodesic is bounded. The condition is inspired by papers of Beck and Chen, where they study billiard trajectories on similar objects [5].

Definition 1.1 (Superdensity)**.** Let ω be a translation surface. We say that ω has *superdense* linear flow Φ_t if there exists a constant $C > 0$ such that for every $T > 0$ where the flow is defined, the segment of the flow Φ_t for $t \in [0, T]$ is within $\frac{C}{T}$ to every point on ω. Equivalently, the segment of the flow Φ_t for $t \in [0, CT]$ is $\frac{1}{T}$-dense on the surface.

Beck and Chen show that a linear flow on a square-tiled surface is superdense if and only if the slope in the associated direction is a badly-approximable number.

We give the following generalization.

Theorem 1.4. Let $\omega \in \Omega_g$ be a translation surface. The linear flow on ω is superdense if the associated Teichmüller geodesic $\{g_t\omega\}_{t\geq 0}$ is bounded in Ω_g. Conversely, if ω has a superdense linear flow, then the diameter of $g_t\omega$ is bounded for all $t \geq 0$.

We prove this result using the *diameter* of the translation surface and *dilatation* of an underlying Teichmüller map to control the quantitative density of the vertical (northward) linear flow. As a corollary, since the diameter function is a proper function on the $SL_2(\mathbb{R})$-orbit closures associated with a lattice surface, we have the following.

Corollary 1.1. Let $\omega \in \Omega_g$ be a lattice surface. The linear flow on ω is superdense if and only if the associated Teichmüller geodesic $\{g_t\omega\}_{t>0}$ is bounded in Ω_g.

1.4. **Cylinder decompositions on infinite translation surfaces.** While most linear trajectories on a translation surface are dense, or dense in a region of the surface, there are a multitude of closed linear trajectories. On a finite translation surface, closed trajectories (that do not pass through a cone point), foliate cylinders on the surface. On lattice surfaces, there are a plethora of cylinder decompositions, and many of these cylinder decompositions can be coupled with parabolic affine self-diffeomorphisms, affine diffeomorphisms whose derivative is a parabolic element in $SL_2(\mathbb{R})$ [17].

The situation for infinite-type translation surfaces is different. An infinite type translation surface has a polygonal representation that can consist of countably infinite polygons in the plane, or polygons with infinitely many sides, or both. Dami Lee and I construct a particular infinite-type translation surface, called an *armadillo tail*, which has interesting features. We place a square, which we denote by $\square_1$, in the first quadrant so that the lower left vertex lies at the origin and all edges are parallel to the axes. For $k \geq 1$, glue the left side of $\square_{k+1}$ to the right side of $\square_k$ so that the bottom edge of all squares lie on the x-axis. We then identify horizontal (vertical, resp.) edges via vertical (horizontal, resp.) translation. Bowman [6] and Degli Esposti–Del Magno–Lenci [10] have also built infinite-type translation surfaces in a similar fashion but allowed rectangles instead of squares; the surface in the former article is known as a "stack of boxes" and the one in the latter, "Italian billiards."

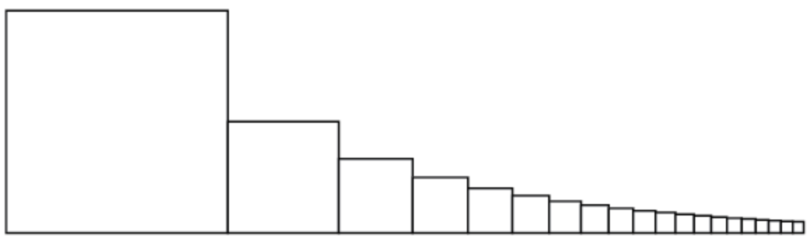

FIGURE 8. Armadillo tail (courtesy Dami Lee)

Armadillo tails are concrete, toy examples that we are using to prod at both the geometric and dynamical properties of finite area, infinite-type translation surfaces *with one wild singularity* (and no other singularities). Wild singularities do not appear on finite-type surfaces: they are singularities which are not finite or infinite angle cone points, but rather, something else entirely. On an armadillo tail, we can see one concretely. Every vertex of the polygonal representation is the same point on the surface. And this point has a curious property: the bottom of each square in the tail is a saddle connection from the wild singularity to itself. And the bottom of a square can be arbitrarily small.

On a particular type of armadillo tail, a *geometric armadillo tail*, where the size of the nth square is $\frac{1}{q}$ times the size of the square to the left, for $q \in \mathbb{N} \setminus \{1\}$, Dami Lee and I construct a cylinder decomposition comprised of infinitely many cylinders. The cylinders can be arbitrarily thin, and the increasingly thin cylinders in the decomposition limit to a single curve on the surface which we call a *spine*. It is noteworthy that the spine is not in any particular cylinder, nor is it the boundary of a cylinder.

Finding cylinder decompositions on a surface is challenging, even for finite-type translation surfaces. Here, we are able to leverage the repeating structure of the surface to inductively construct cylinders [24].

The cylinder decomposition has another curious property. "Truncate" this geometric armadillo tail by taking the first n-squares in the surface, and identify edges as before but also glue the right

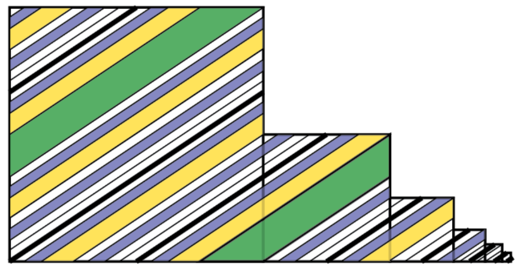

FIGURE 9. Partial cylinder decomposition on a geometric Armadillo Tail (courtesy Dami Lee)

side of the nth square to the left side of the first square. On any truncated surface, the truncated cylinder decomposition turns out to be a cylinder decomposition, and there is a parabolic affine automorphism on the truncated surface which preserves this cylinder decomposition. However, in the "limit", the infinite-type armadillo tail, there is no parabolic affine automorphism that preserves the cylinder decomposition.

There is an open question regarding whether or not there exists an finite area, infinite-type translation surface with a Veech group that is a lattice in $SL_2(\mathbb{R})$ (see [11]). Recall that the Veech group is the collection of derivatives of affine automorphisms. One might think that a particular geometric armadillo tail is a candidate, but it seems as though the Veech group may be $\mathbb{Z}$.

Question 1.6. What is the Veech group of a geometric armadillo tail?

If the Veech group is a lattice, and not $\mathbb{Z}$, then the work of Treviño shows that the linear flow is ergodic [35]. If the Veech group is not a lattice, the situation may be more complicated.

Question 1.7. Is the linear flow ergodic? If so, what ergodic measures are supported on the surface?

1.5. **The gap distribution of saddle connections.** The boundary of every cylinder on a translation surface is comprised of saddle connections, linear segments whose endpoints are singularities. If we fix a singularity, the number of directions from the singularity in which there is a saddle connection is infinite.

One can ask about the *gaps* between saddle connections in the following sense. Fix a radius R and enumerate the set of saddle connections of length at most R. There will be finitely many, and we can compute the gaps between the direction. Now, let R grow and the gaps between the saddle connections will decrease as more appear. If we renormalize the gaps by the rate at which the saddle connections are appearing, do the renormalized gaps converge to something?

Boca, Cobeli, and Zaharescu recognized that the *slope gaps* of saddle connections (gaps between the slopes of the saddle connections, not the directions) on the torus, when correctly renormalized, converge to a function called Hall's distribution. Rather, Boca, Cobeli, and Zaharescu were studying gaps in the set of Farey fractions,

$$\mathcal{F}(Q) := \{\text{reduced fractions } \frac{p}{q} : (q, p) \in \mathbb{Z}^2, 0 < q \leq Q\}$$

as the denominator of the fraction grows. The Farey fractions $\mathcal{F}(Q)$ are also precisely the slopes of the saddle connections on the torus. In this way, it can be seen that the *slope gap* distribution of the square torus is the gap distribution of the Farey fractions.

Athreya and Cheung recognized that one can observe convergence of the renormalized slope gaps to Hall's distribution using the horocycle flow in the moduli space of tori. Tariq Osman, Jane Wang, and I have effectivized these results.

Theorem 1.5 (Effective gap distribution)**.** Let $F(x)$ be the slope gap distribution of the square torus with one marked point, which is also the limiting gap distribution of the Farey fractions. Let $\mathbb{G}_R$ be the renormalized gaps of the slopes Λ_R for the square torus, or the renormalized gaps of the Farey fractions of denominator $\leq R$. Then, there exists a constant $C > 0$ such that for any $0 \leq a < b < \infty$,

$$\left| \frac{|\mathbb{G}_R \cap (a,b)|}{N(R)} - \int_a^b F(x)\,dx \right| \leq C \log(R) R^{-\frac{1}{3}}$$

.

The gap distribution is also understood for generic lattice surfaces [21]. We were able to effectivize the results as well.

Theorem 1.6 (Effective gap distribution (lattice surface))**.** Let $F(x)$ be the slope gap distribution of the lattice surface (X,ω). If (X,ω) has a vertical saddle connection, then for any $0 \leq a < b < \infty$, and any $\varepsilon > 0$,

$$\left| \frac{|\mathbb{G}_R \cap (a,b)|}{N(R)} - \int_a^b F(x)\,dx \right| \leq \begin{cases} C \log(R) R^{-\frac{1}{4n}-\varepsilon} & \text{if } \Gamma \text{ is tempered} \\ C(s) R^{-(\frac{1}{4n}-\varepsilon)(1-s)} & \text{if } \Gamma \text{ is non-tempered.} \end{cases}$$

where the constant C depends on the choice of lattice surface, and if Γ is not tempered, s, where $\frac{1-s^2}{4}$ is the bottom of the spectrum of the hyperbolic Laplacian on $\mathbb{H}/\Gamma$. If (X,ω) does not have a vertical saddle direction, then for any $0 \leq a < b < \infty$,

$$\left| \frac{|\mathbb{G}_R \cap (a,b)|}{N(R)} - \int_a^b F(x)\,dx \right| \leq \begin{cases} C \log(R) R^{-\frac{1}{9}} & \text{if } \Gamma \text{ is tempered} \\ C(s) R^{-\frac{1}{9}(1-s)} & \text{if } \Gamma \text{ is non-tempered.} \end{cases}$$

For our purposes, a *tempered lattice* is one such that $\frac{1}{4}$ is the bottom of the spectrum of the hyperbolic Laplacian on $\mathbb{H}/\Gamma$, noninclusive of the 0, whereas a non-tempered lattice is one such that the bottom of the spectrum of the hyperbolic Laplacian on $\mathbb{H}/\Gamma$ is strictly less than $\frac{1}{4}$. The spectral parameter s in Theorem 1.6 quantifies the spectral gap.

The proof of these theorems rests on an effective equidistribution theorem for the intersection points of a family of long horocycles with a particular Poincaré section Ω which is transverse to the horocycle flow. If $\Gamma \subset SL_2(\mathbb{R})$ is the Veech group of (X,ω), then $\Omega \subset SL_2\mathbb{R}/\Gamma$ is the set of translation surfaces in the $SL_2(\mathbb{R})$ orbit of (X,ω) that has a short (length ≤ 1) horizontal saddle connection.

Proving this equidistribution result requires using known results for the effective equidistribution of long horocycles in $SL_2(\mathbb{R})/\Gamma$. This leads to the following question. In order to extend this result to a larger class of translation surfaces, one would need to generalize this equidistribution result.

Question 1.8. Can one prove an equidistribution statement for long closed horocycles on general affine invariant manifolds? This seems perhaps too broad, but one can start to understand obstructions by studying the characters of the *-algebra of $SO(2)$-invariant measures on $SL_2(\mathbb{R})$ (recall that horocycles are closely approximated by circles). However, this algebra is not commutative. The author is unsure if anyone has initiated a systematic study of these characters.

Our result also depends on a lattice point counting result of Burrin, Nevo, Ruhr, and Weiss ([7], see Theorem 2.7) which applies to counting lattice points in general star-shaped domains as these domains dilate. This lattice point counting result is a consequence of the machinery in work of Gorodnik and Nevo [15].

Question 1.9. Can we count lattice points in triangular domains more efficiently than in general star-shaped domains?

A positive answer to this question could improve the rate in the case of a lattice surface with a vertical saddle connection.

2. Metric Graphs

The following is from ongoing work with Junaid Hasan and Farbod Shokrieh. In 2006, Mikhalkin and Zharkov proved that there is a one-to-one correspondence between metric graphs and compact tropical curves, which established a connection between the study of metric graphs and tropical geometry. Tropical geometry is analogous to complex geometry in some surprising ways. For example, there is a rather mysterious Riemann-Roch theorem for graphs that was proven by Baker and Norine [4] and subsequently extended to tropical curves [14], [25]. The only known proof of this theorem relies on Dhar's burning algorithm, which is purely combinatorial. We ask whether or not an analytic proof of Riemann-Roch can be constructed. Classically, such proofs in complex geometry leverage properties of the Laplacian. However, in the case of the tropical Riemann-Roch theorem, it is unclear *what functions* the Laplacian would be acting on, leading to a deeper question about what sheaf we should be studying.

Our interest is in a particular set of functions (or rather forms): we plan to use the theory of Chambert-Loir and Ducros [8] on tropical forms and currents. In 1973, Ray and Singer computed analytic torsion on complex manifolds [29], and so we will probe the deeper question by computing a variant of *analytic torsion* on metric graphs. My focus is the role that harmonic analysis plays.

2.1. Regularized determinants, heights, and analytic torsion on metric graphs.

Analytic torsion is an invariant on a Riemannian manifold that depends on the underlying topology, a chain complex of forms on the surface, and a representation of the fundamental group of the surface. The definition of analytic torsion relies on the following definition of the determinant of the Laplacian, which we call a *regularized determinant*.

$$\det(\Delta) = e^{-\zeta'(0)}$$

where $\zeta(s) = \sum_{\lambda\in\sigma} \frac{1}{\lambda^s}$ is the zeta function packaging the spectrum of the Laplacian, σ. Here, we should note that we are using an analytic continuation of the zeta function to a meromorphic function on $\mathbb{C}$, and the function takes a value at 0. Hence, the function is holomorphic in a neighborhood of zero, and $\zeta'(0)$ makes sense.

It is not hard to check that this definition generalizes the usual determinant of a linear transformation on a finite dimensional vector space. The usual determinant is the product of the (finitely many) eigenvalues λ_i and

$$\prod_{i=1}^{n} \lambda_i = e^{-\frac{d}{ds}\left(\sum_{i=1}^{n} \frac{1}{\lambda_i^s}\right)\Big|_{s=0}}.$$

The derivative of the zeta function, or the logarithm of the determinant, is of independent interest since it connects to the *theory of heights* [30].

2.2. The circle.

Consider the circle $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, which we think of as a simple example of a graph. We can compute eigenvalues of the operator directly by solving $\Delta f = \lambda f$. The non-zero spectrum σ is $\{n^2 : n \in \mathbb{Z}\}$.

We package the non-zero spectrum in a zeta-function, $\zeta(s)$, which has a nice connection to the Riemann zeta function $\zeta_R(s)$:

$$\begin{aligned}\zeta(s) &= \sum_{\lambda\in\sigma}\frac{1}{\lambda^s}\\ &= \sum_{n\in\mathbb{Z}}\frac{1}{n^{2s}}\\ &= 2\sum_{n\in\mathbb{N}}\frac{1}{n^{2s}}\\ &= 2\zeta_R(2s)\end{aligned}$$

This means $\zeta'(s) = 4\zeta_R'(2s)$, hence $\zeta'(0) = 4(-\frac{1}{2}\log 2\pi) = -2\log 2\pi$. This gives the following value for the regularized determinant:

$$\det(\Delta) = e^{-\zeta'(0)} = (2\pi)^2 = 4\pi^2$$

To compute the analytic torsion, we pick a representation of the fundamental group $\mathbb{Z}$ and use the deRham cohomology on S^1. However, this is where our curiosities diverge from the classical set-up: what do our forms look like on the graph?

2.3. **General metric graphs.** For the case of a general graph, there is an additional complication in the spectral theory. Similar to translation surfaces, a generic metric graph may have more than one self-adjoint Laplacian. The choice of self-adjoint Laplacian is intimately tied to the sheaf we are interested in. For example, Kurasov and Sarnak have recently computed trace formulae for metric graphs. They define the Laplacian on a set of sufficiently differentiable functions which are continuous at vertices and the sum of the incoming slopes at vertices is zero [22].

There is another question lingering for general metric graphs. Ray and Singer first defined analytic torsion with the suspicion that it was an analytic version of R-Torsion, an invariant defined on topological spaces, the difference being that R-Torsion is an invariant defined through the homology whereas analytic torsion is defined through the de Rham cohomology [28]. Several years later, Cheeger and Müller independently proved that this is true [9], [26].

Question 2.1. Is R-Torsion equivalent to analytic torsion on metric graphs? For what sheaf? Does this give any indication as to the kinds of duality to be expected between homology and cohomology in tropical geometry?

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