## CYLINDER DECOMPOSITIONS ON GEOMETRIC ARMADILLO TAILS

## DAMI LEE AND JOSH SOUTHERLAND

ABSTRACT. We study a particular type of finite-area, infinite-type translation surfaces, and find an explicit cylinder decomposition on these surfaces which do not manifest on finite-type translation surfaces.

### 1. INTRODUCTION AND DEFINITIONS

A translation surface is a (countable) collection of polygons in the plane where all edges are paired with an edge of equal length by translation such that inward pointing normal vectors on each edge point in opposite directions after identification. Finite-type translation surfaces are collections of finitely many polygons with only finitely many edges, whereas infinite-type translation surfaces allow for constructions containing countably many polygons allowing infinitely many edges. See, for example, [4].

In this article, we define a particular infinite-type translation surface to study, which we call an *armadillo tail surface*, or *armadillo tail*. For computational purposes, we place a square, which we denote by  $\Box_1$ , so that the lower left vertex lies at the origin and all edges are parallel to the axes. For  $k \ge 1$ , glue the left side of  $\Box_{k+1}$  to the right side of  $\Box_k$  so that the bottom edge of all squares lie on the *x*-axis. We denote the side length of  $\Box_k$  by  $l_k$ , and assume that  $(l_k)$  is a strictly decreasing sequence. We then identify horizontal (vertical, resp.) edges via vertical (horizontal, resp.) translation. Bowman [2] and Degli Esposti–Del Magno–Lenci [3] have also built infinite-type translation surfaces in a similar fashion but allowed rectangles instead of squares; the surface in the former article is known as a "stack of boxes" and the one in the latter, "Italian billiards."

The following are examples of finite-area armadillo tail surfaces.

**Example 1.1.** (1) The armadillo tail surface where  $l_k = r^{k-1}$ , for  $r \in (0, 1)$ , which we call a *geometric armadillo tail* with parameter *r*. It is bounded and has finite area,  $\frac{1}{1-r^2}$ .

(2) The *harmonic armadillo tail* where  $l_k = \frac{1}{k}$  (Figure 1). While the surface is not bounded in the horizontal direction, its area is  $\zeta(2) = \frac{\pi^2}{6}$ , finite.



FIGURE 1. The harmonic armadillo tail surface

We attain a finite translation surface  $\bigcup_{k=1}^{n} \Box_k$  which we call the *truncated armadillo tail* where we make the same identifications as above, but now we identify the right edge of  $\Box_n$  with the bottom segment of the left edge of  $\Box_1$ . We denote the truncated armadillo tail by  $X_n$ .

Without loss of generality, we assume that  $l_1 = 1$ . With horizontal (vertical, resp.) edges being identified via vertical (horizontal, resp.) translation, the resulting translation surface is an infinite genus surface (infinite connected sum of square tori) with one (wild) singularity. For background on wild singularities, see [2], [4], [10]. The wild singularity appears infinitely many times in the polygonal representation of the surface: each vertex of the infinite-sided polygon is the same point on the surface.

Armadillo tails are concrete, toy examples that we use for prodding at both the geometric and dynamical properties of finite-area, infinite-type translation surfaces with one wild singularity (and no other singularities). In what follows, our focus is on a purely geometric construction: a cylinder decomposition on this surface. Finding cylinder decompositions on a surface is challenging, even on finite translation surfaces. Here, we are able to leverage the structure of the surface to inductively construct cylinders.

A *cylinder* is a closed subspace of the surface whose interior is foliated by homotopic closed straight-line trajectories, and whose boundary consists of *saddle connections*, line segments whose endpoints coincide with a singular point and none in the interior. A closed geodesic in the interior of a cylinder is called a *waist curve*. The *circumference* of a cylinder is the length of a closed straight-line trajectory, and the *width* of a cylinder is the distance between the bounding saddle connections. A *cylinder decomposition* is the closure of a union of possibly infinitely many cylinders whose waist curves are in the same direction and which covers the surface. Further, we require that each cylinder in the cylinder decomposition only intersect another cylinder at most along a saddle connection. The closure of the union of cylinders may contain a line segment that is not in any cylinder which we call a *spine*. If a spine is made of a single saddle connections, we call it a *rigid spine*. If a cylinder decomposition has no spine, we say it is a *complete cylinder decomposition*.

As an example, consider the cylinder decomposition C of an armadillo tail in the horizontal direction, given our choice of polygonal representation of the surface. See Figure 2.



FIGURE 2. A cylinder decomposition of an armadillo tail in the horizontal direction

Label the cylinders numerically from top to bottom,  $cyl_k$ , and observe that each cylinder in the cylinder decomposition becomes longer and thinner, eventually limiting to a concatenation of infinitely many saddle connections at the base of the polygonal representation. This may appear to be a flexible spine, but each of these saddle connections is a boundary component of a cylinder, namely the "top" of a cylinder, since the bottom saddle connections are identified to the saddle connections appearing at the top of each square. Indeed, this is a complete cylinder decomposition.

There is another cylinder decomposition intimately related to the above cylinder decomposition, what we call the *orthogonal cylinder decomposition*,  $C^{\perp}$ . See Figure 3. Then,  $cyl_1^{\perp}$ , the first cylinder in  $C^{\perp}$  has width equal to the circumference of  $cyl_1$  in C. The circumference of  $cyl_1^{\perp}$  is equal to the sum of all of the widths of the cylinders in the original cylinder decomposition. Similarly,  $cyl_2^{\perp}$  has width equal to the height of  $cyl_2$  less the height of  $cyl_1$ . The circumference of  $cyl_2^{\perp}$  is the sum of all of the widths of the cylinders in C less the width of  $cyl_1$ . Each subsequent cylinder is defined in the same manner.



FIGURE 3. A cylinder decomposition of an armadillo tail in the vertical direction

**Remark 1.2.** Observe that the modulus of each cylinder in the orthogonal cylinder decomposition is 1. This implies that there exists an affine diffeomorphism of the surface  $\phi$  such that  $D\phi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ , where  $D\phi$  is an element in the Veech group. See Appendix D in [5] for additional information. Observe that  $D\phi$  is a parabolic element in  $SL_2(\mathbb{R})$ , where the eigendirection of  $D\phi$  corresponds to the direction of the cylinder decomposition. We will refer to an affine diffeomorphism whose derivative is a parabolic element as a *parabolic affine diffeomorphism*. For a finite-type translation surface, a parabolic affine diffeomorphism is an example of a reducible element in the mapping class group of the underlying topological surface.

1.1. **Main results.** There is another less obvious cylinder decomposition on the surface which does not appear in the orbit of this horizontal cylinder decomposition (orbit of the group of affine diffeomorphisms of the surface). In the same way that the above (horizontal) cylinder decomposition is comprised of an infinite number of cylinders limiting to the base of the polygonal representation, this cylinder decomposition also contains an infinite number of cylinders becoming thinner and thinner. However, the distinction is that this is not a complete cylinder decomposition - there is a rigid spine.

**Theorem 1.3.** There exists a cylinder decomposition with a rigid spine on any geometric armadillo tail of parameter  $\frac{1}{q}$ ,  $q \in \mathbb{N} \setminus \{1\}$ . Moreover, there is no parabolic affine diffeomorphism of the surface that fixes this cylinder decomposition.

The existence of the rigid spine is interesting in its own right.

**Proposition 1.4.** Assume that a surface has a set of rigid spines which are a subset of saddle connections. Then, if the set of rigid spines is discrete, the Veech group is discrete.

This follows from the fact that a rigid spine must be mapped to a rigid spine under an affine diffeomorphism of the surface. Then, the same argument that shows that a discrete set of saddle connections implies that the Veech group of a translation surface is discrete will work here. For this argument, see for example, [8]. In a subsequent work, we construct various examples of rigid spines on infinite-type translation surfaces and draw further connections to affine diffeomorphisms of these surfaces [9].

The proof of the Theorem 1.3 is constructive. In Section 2, we identify a core curve in a special direction, which turns out to be a rigid spine of a cylinder decomposition. This curve wraps around every torus in the surface. Then we construct a cylinder which turns out to be the widest cylinder in the cylinder decomposition. In Section 3, we inductively construct a collection of saddle connections which turn out to frame all of the cylinders in our cylinder decomposition. We use two techniques to do this, one involving an infinite interval exchange transformation (IET). See [12] for a description of IETs. In Section 4, we construct the cylinder decomposition by defining a (discontinuous) map which pushes a cylinder to a subset of the next widest cylinder in the decomposition. We call the subset of a cylinder using a circle rotation argument. Indeed, the endpoints of the partial cylinder correspond to periodic points of a circle rotation. In Section 5, we compute the modulus and area of each cylinder in the cylinder decomposition. In Section 6, we show that there cannot be a parabolic element that stabilizes the cylinder decomposition that we construct. Moreover, we note that the orthogonal cylinder decomposition to this cylinder decomposition *does not exist*.

It seems feasible to extend our methods to the case of  $r \in \mathbb{Q}$ , provided one can find enough cylinders to start the induction process in Section 4. Moreover, the induction process may involve fixed points of a finite-type interval exchange transformation in lieu of a circle rotation.

1.2. **Related work.** Bowman studies the "geometric limit" of finite translation surfaces converging to an infinite-type translation surface [1]. Along these lines, we can think of an armadillo tail as the limit of finite-type translation surfaces. Indeed, consider the truncated surface  $X_n$  of a geometric armadillo tail. The cylinder decomposition in the special direction persists for all surfaces in this sequence. Moreover, for all finite surfaces in the sequence, there is a parabolic element that preserves both this cylinder decomposition and another that preserves the orthogonal cylinder decomposition. However, in the limit, the cylinders converge to a cylinder decomposition, but the parabolic affine diffeomorphism does not converge to any sensible affine map on the surface.

For particular directions on an armadillo tail, one can use Treviño's work, Theorem 3 in [11], to explore ergodicity of the linear flow in that direction. However, there does not appear to be a (Veech–Treviño) dichotomy in which each direction is either periodic or ergodic: the Veech group appears to be  $\mathbb{Z}$  and not a lattice, but this is unclear. It is an interesting question as to what ergodic

measures are supported on an armadillo tail (geometric or otherwise). A generalization of a Veech dichotomy of this flavor was done for infinite staircase surfaces by Hooper, Hubert, and Weiss [6].

Lastly, there is an open question regarding whether or not there exists a finite-area, infinite-type translation surface with a Veech group that is a lattice in  $SL_2(\mathbb{R})$  (see [4]). It seems reasonable to think that there may exist an armadillo tail whose Veech group is a lattice, but this is unclear.

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## 2. The spine and the first cylinder (Or a particular direction on geometric Armadillo tails)

The following is a key theorem in which we identify a closed saddle connection which turns out to be a (limiting) rigid spine of a cylinder decomposition on a certain family of armadillo tails. Every armadillo tail is an infinite connected sum of tori; this particular saddle connection wraps around each torus. Note that the following theorem requires no assumption on the parameter *r*. By  $\frac{1}{2-r}$ -direction, we mean the direction with slope  $\frac{1}{2-r}$  relative to our polygonal representation. For example, see Figure 4.



FIGURE 4. A geometric armadillo tail (r = 4/5) with a trajectory of slope 5/6

**Theorem 2.1.** On a geometric armadillo tail with  $r \in (0, 1)$ , there exists a closed saddle connection in the  $\frac{1}{2-r}$ -direction that intersects every torus.

*Proof.* We refer to the top (horizontal) edge of a square by the *roof* and the right (vertical) edge of a square that is identified to a segment on the *y*-axis by the *portal*.

We start from the origin, the lower left vertex of  $\Box_1$ . Since  $r < \frac{1}{2-r} < 1$ , the straight line of slope  $\frac{1}{2-r}$  through the origin hits the portal of  $\Box_1$  at point  $(1, \frac{1}{2-r})$ . By identification with the *y*-axis (the left edge of  $\Box_1$ ), the trajectory continues and hits the roof of  $\Box_1$  at (1 - r, 1). By identification with the bottom edge of  $\Box_1$ , the trajectory continues from (1 - r, 0) and hits the roof of  $\Box_2$  at (1 + r(1 - r), r). The trajectory partitions both roofs with a fixed ratio. Due to similarity in consecutive squares, it continues to partition subsequent roofs with the same ratio. Needless to say, the trajectory "wraps around every torus" without hitting any vertex.

This saddle connection has an interesting topological feature.

**Proposition 2.2.** On a geometric armadillo tail with parameter *r*, the saddle connection constructed above is a non-separating simple closed curve.

*Proof.* If we color the surface on one side of the saddle connection, we color the entire surface.  $\Box$ 

Via renormalization (under the action of  $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ , see Remark 1.2), one can consider the trajectory with slope  $\frac{1}{2-r} - 1 = \frac{r-1}{2-r}$  direction. Starting from the upper left vertex at (0, 1), the trajectory hits the portal on  $\Box_1$  at  $(1, \frac{1}{2-r})$ . By identification, continuing from  $(0, \frac{1}{2-r})$ , the trajectory does not hit any roof or portal and tends to  $(\frac{1}{1-r}, 0)$ , hence producing a saddle connection. In fact, given our polygonal representation, any trajectory starting from (0, 0) with slope  $\frac{1}{2-r} + \mathbf{n}$ , for any  $\mathbf{n} \in \mathbb{N}$ , (or (0, 1) with slope  $\frac{1}{2-r} - \mathbf{n}$ , for  $\mathbf{n} \in \mathbb{N}$ ) yields a saddle connection that goes through every torus. We will see that this saddle connection is the rigid spine of a cylinder decomposition.

In the following theorem, we show that for geometric armadillo tails with  $r = \frac{1}{q}$  for  $q \in \mathbb{N} \setminus \{1\}$ , there exists not only a saddle connection but a cylinder in the  $\frac{1}{2-r}$ -direction. See Figure 5. We call this cylinder cyl<sub>1</sub>, and the existence of this cylinder will be part of the base case for the induction in Section 4.



FIGURE 5. The cylinder cyl<sub>1</sub> on a geometric armadillo tail with r = 1/2 (left) and 1/3 (right)

**Theorem 2.3.** Given a geometric armadillo tail with parameter  $r = \frac{1}{q}$ , there exists a cylinder in the  $\frac{1}{2-r}$ -direction which lies entirely in  $\Box_1 \cup \Box_2$ .

There are three parts to this proof. First, we show that there is a saddle connection (in the  $\frac{1}{2-r}$ -direction) that lies entirely in  $\Box_1 \cup \Box_2$ . Secondly, we will show that there is another saddle connection in  $\Box_1 \cup \Box_2$  parallel to the first one and of the same length. In the third step, we will show that there is no saddle connection between the two saddle connections, and consequently that the interstitial space is foliated by closed geodesics, hence yielding a cylinder.

*Proof.* Note, for  $q = \mathbb{N} \setminus \{1\}$ , the slope of the trajectory is  $\frac{1}{2-r} = \frac{q}{2q-1}$ .

**Step 1.** We show that the trajectory from (1,0) with slope  $\frac{q}{2q-1}$  stays entirely in  $\Box_1 \cup \Box_2$ .

Start from (1,0) we hit the portal of  $\Box_2$  at  $\left(1+\frac{1}{q},\frac{1}{2q-1}\right)$ , hence the first time the trajectory hits the *y*-axis is at  $\left(0,\frac{1}{2q-1}\right)$ . We continue and hit  $\left(1,\frac{1+q}{2q-1}\right)$ , which lies in the portal of  $\Box_1$  since  $\frac{1}{q} < \frac{1+q}{2q-1} < 1$ . It is then identified to  $\left(0,\frac{1+q}{2q-1}\right)$ . Note that if we keep hitting the portal, the *n*th time the trajectory hits the vertical axis is at  $\left(0,\frac{1}{2q-1}+\frac{(n-1)q}{2q-1}-\lfloor\frac{1}{2q-1}+\frac{(n-1)q}{2q-1}\rfloor\right)$ . In fact, since  $\frac{1}{2q-1} < \frac{1}{q} < \frac{2}{2q-1}$ , we always hit the portal unless  $\frac{1}{2q-1} + \frac{(n-1)q}{2q-1} - \lfloor\frac{1}{2q-1} + \frac{(n-1)q}{2q-1}\rfloor = \frac{1}{2q-1}$ . Pick n = 2q - 2, then the trajectory hits the singularity at (1, 1). In other words, the trajectory goes through  $\Box_2$  exactly once at the beginning and stays in  $\Box_1$  until it hits (1, 1). The saddle connection that we constructed will serve as the "bottom" boundary saddle connection of  $cyl_1$ . Furthermore, this is the first time the trajectory hits a singularity. This follows from the fact that gcd(q, 2q - 1) = 1. The trajectory hits the *y*-axis at points  $\left\{ (0, y) : y = \frac{(n-1)q+1}{2q-1} - \lfloor \frac{(n-1)q+1}{2q-1} \rfloor, n = 1, \dots, 2q - 3 \right\}$ . Note that this set can be rearranged as

$$\left\{\frac{1}{2q-1}, \frac{1+q}{2q-1}, \frac{1+2q}{2q-1}, \dots, \frac{1+(2q-3)q}{2q-1} \equiv 1\right\}.$$

Note that if we continue to add  $\frac{q}{2q-1}$ , we hit  $\frac{1+(2q-2)q}{2q-1} \equiv \frac{q}{2q-1}$  and  $\frac{1+(2q-1)q}{2q-1} \equiv \frac{1}{2q-1}$ , which brings us back to the beginning of the sequence. In other words, the trajectory wraps around  $\Box_1$  exactly 2q - 3 times hitting the *y*-axis at  $\left\{ \left(0, \frac{i}{2q-1}\right) \right\}_{i=1, \neq q}^{2q-2}$ . Alternatively, notice that once the trajectory optors  $\Box_1$  we are served with division of the sequence.

Alternatively, notice that once the trajectory enters  $\Box_1$ , we can encode the hitting points on the portal (which are identified to the left side of  $\Box_1$ ) via a circle rotation that arises as a section of the linear flow with slope  $\frac{q}{2a-1}$ . We use this perspective in Steps 2 and 3 below.

**Step 2.** Next, we construct a saddle connection which will be the "top" boundary saddle connection. To do this, we will take the bottom saddle connection and show that if we shift the saddle connection vertically (in the polygonal representation) by  $\frac{q-1}{q(2q-1)}$ , that we find another saddle connection. A consequence of Step 3, which follows, is that  $\frac{q-1}{q(2q-1)}$  is the *skew-width* of this cylinder, the vertical distance (with respect to the polygonal representation) between the two saddle connections. The skew-width is formally defined in Section 4.

Take the set of points where the bottom saddle connection hits the *y*-axis and add  $\frac{q-1}{q(2q-1)}$ :

$$\left\{\frac{i}{2q-1}\right\}_{i=1,\neq q}^{2q-2} + \frac{q-1}{q(2q-1)} = \left\{\frac{(i+1)q-1}{q(2q-1)}\right\}_{i=1,\neq q}^{2q-2}$$

where a set + number denotes adding the number to each element of the set. We split this set into three cases: (1) i = 1, ..., q - 2, (2) i = q - 1, and (3) i = q + 1, ..., 2q - 2.

Let  $T : [0,1]/_{\sim} \to [0,1]/_{\sim}$  be a circle rotation where  $T(x) = x + \frac{q}{2q-1}$ . Note that the circle rotation is a section of the linear flow on  $\Box_1$ , provided we never enter  $\Box_2$ . We can guarantee that the linear flow does not enter  $\Box_2$  provided the iterates of the circle rotation are greater than or equal to  $\frac{1}{q}$ . Observe that every point in the set  $\left\{\frac{(i+1)q-1}{q(2q-1)}\right\}_{i=1,\neq q}^{2q-2}$  is greater than or equal to  $\frac{1}{q}$ . Moreover, the image of these points under *T* is always greater than  $\frac{1}{q}$ , except for the image of the point corresponding to case (2): i = q - 1. We will address this when it arises.

First, observe that *T* maps points corresponding to case (1) to points corresponding to case (3). Indeed, take points in case (1), (i = 1, ..., q - 2), and apply *T*. We have

$$T\left(\frac{(i+1)q-1}{q(2q-1)}\right) = \frac{(i+1+q)q-1}{q(2q-1)} \in \left\{\frac{(j+1)q-1}{q(2q-1)}\right\}_{j=q+1}^{2q-1}.$$

Next, observe that the points in case (3) map to points in case (1), with one exception, in which case the image of point corresponds to the point in case (2). Take points in case (3), (i = q + 1, ..., 2q - 2) and apply *T*. For all cases except i = 2q - 2 we have

$$T\left(\frac{(i+1)q-1}{q(2q-1)}\right) = \frac{(i+1+q)q-1}{q(2q-1)} \equiv \frac{(i-q+2)q-1}{q(2q-1)} \in \left\{\frac{(j+1)q-1}{q(2q-1)}\right\}_{j=1}^{q-2}.$$

If i = 2q - 2, then we have  $T\left(\frac{(2q-1)q-1}{q(2q-1)}\right) = \frac{q^2-1}{q(2q-1)} \in \left\{\frac{(i+1)q-1}{q(2q-1)}\right\}_{i=q-1}$ , which corresponds to the point in case (2).

Lastly, consider the point in case (2), where i = q - 1. We have

$$T\left(\frac{q^2-1}{q(2q-1)}\right) = \frac{2q^2-1}{q(2q-1)} \equiv \frac{q-1}{q(2q-1)} < \frac{1}{q}.$$

Since the trajectory hits the right side of  $\Box_1$  below the portal, we continue into  $\Box_2$  and hit  $\left(1 + \frac{1}{q}, \frac{1}{q}\right)$ , the singularity at the upper right vertex of  $\Box_2$ .

Now, observe that if we begin with i = 1, the sequence of iterates of *T* will include every point in case (1) and case (3), and then hit the point in case (2). In other words, the suspension of the linear flow is a closed saddle connection that passes through  $\left(0, \frac{1}{q}\right)$  that lies entirely in  $\Box_1$  and  $\Box_2$ .

**Step 3.** Lastly, we show that between these two saddle connections there is no other saddle connection in with slope  $\frac{q}{2q-1}$ . In other words, there are no saddle connections that hit the vertical axis between  $\left(0, \frac{1}{2q-1}\right)$  and  $\left(0, \frac{1}{q}\right)$ .

Define the intervals along the *y*-axis with *y*-coordinates in  $(\frac{i}{2q-1}, \frac{i}{2q-1} + \frac{q-1}{q(2q-1)})$ , for  $i \in \{1, 2, ..., 2q-2\} \setminus \{q\}$ . The lower bound in each interval coincides with an intersection of the (bottom) saddle connection constructed in step 1 with the *y*-axis. Similarly, the upper bound coincides with an intersection of the (top) saddle connection constructed in Step 2 with the *y*-axis.

First, observe that the intervals do not contain a singularity. The only appearances of the wild singularity along the *y*-axis for  $y > \frac{1}{2q-1}$  occur for  $y = \frac{1}{q}$  and y = 1, neither of which land inside any of the intervals.

Now, fix any  $0 < \varepsilon < \frac{q-1}{q(2q-1)}$  and consider the collection of points  $(0, y_i)$  for  $y_i = \frac{i}{2q-1} + \varepsilon$ , for  $i \in \{1, 2, ..., 2q-2\} \setminus \{q\}$ . (There is one point in each of the intervals.) By applying the map *T* to the point in the interval corresponding to i = 1, we see that the image is the point in the collection corresponding to the interval i = q + 1. The argument is the same as the one given in Step (2). We continue applying the map *T* until we reach the last interval in the set,  $(0, y_{q-1})$ . Here the image of the circle rotation contains the singular point infinitely many times, but this is because the circle rotation no longer applies. Indeed,  $T(y_{q-1}) < \frac{1}{q}$  which means that the suspended flow is actually entering  $\Box_2$ . (This is identical to the situation described in case (2) in step 2.) If we flow in the linear direction from  $(0, y_{q-1})$ , we hit the portal in  $\Box_2$  at  $(1 + \frac{1}{q}, y_1)$ , which means that the linear flow starting at  $(0, y_1)$  is a closed geodesic.

Moreover, since the choice of  $\varepsilon$  allows for any point in the interval, the suspension of these intervals consists of closed geodesics.

In conclusion, on a geometric armadillo tail with parameter  $r = \frac{1}{q}$ , there exists a cylinder that lies entirely in  $\Box_1$  and  $\Box_2$ . We call this cylinder cyl<sub>1</sub>.

**Remark 2.4.** The assumption that  $r = \frac{1}{q}$  cannot be removed. For example, if r = 2/3, the saddle connection starting from (1, 0) in the  $\frac{1}{2-r}$ -direction is not contained in  $\Box_1$  and  $\Box_2$ , but also passes through  $\Box_3$ .

## 3. FRAMING (OR FINDING THE BOTTOM SADDLE CONNECTION OF EACH CYLINDER BY INDUCTION)

In this section, we construct concatenations of saddle connections that will bound the cylinders in our cylinder decomposition. We will label them  $bsc_k$ , and in Section 4, these sets will be realized

as the **b**ottom boundary saddle connections of the *k*th widest cylinder in the cylinder decomposition. To reduce the notational complexity, and since all of the following statements fix q, we suppress the dependence on q.

We define  $bsc_1$  as a single saddle connection: the saddle connection constructed in Step 1 of Theorem 2.3. However, we construct  $bsc_2$  as a concatenation of two saddle connections, one being the saddle connection constructed in Step 2 of Theorem 2.3 (the top of the  $cyl_1$ ), the other being a saddle connection that does not bound  $cyl_1$ . This second saddle connection will be realized as the bottom boundary of the part second widest cylinder which does not run adjacent to  $cyl_1$ . We call this second saddle connection  $bsc'_2$ , and construct it in Lemma 3.1. In fact,  $bsc_k$  will always be a concatenation of two saddle connections.

Assuming Lemma 3.1, observe that we can write  $bsc_2 = bsc_1 + \frac{q-1}{q(2q-1)} \oplus bsc'_2$ , where  $\oplus$  means concatenate, and addition means shifting the saddle connections in  $bsc_1$  vertically by  $\frac{q-1}{q(2q-1)}$ . To shift saddle connections vertically, we remove the singular point from the saddle connections, then vertically shift, and then take the closure. Note that a vertical shift is not well-defined at a singular point.

We will use this observation to inductively construct  $bsc_k = bsc_{k-1} + \frac{q-1}{q^{k-1}(2q-1)} \oplus bsc'_k$ . We note that  $\frac{q-1}{q^{k-1}(2q-1)}$  will be the skew-width of the (k-1)th cylinder, which is the vertical distance with respect to the polygonal representation between the saddle connections bounding the cylinder. In Subsection 3.1, we will construct  $bsc'_k$  by carefully tracking segments along the linear flow. In Subsection 3.2, we will vertically shift each  $bsc'_{k-1}$  by  $\frac{q-1}{q^{k-1}(2q-1)}$ , and extend this along the linear flow, we will use an interval exchange transformation.

3.1. **Constructing** bsc'<sub>k</sub>. In the following lemma, for every  $q \in \mathbb{N} \setminus \{1\}$ , we construct bsc'<sub>2</sub>. We use the linear flow with slope  $\frac{q}{2q-1}$ , but the reader may also interpret the work as identifying fixed points of an IET arising from a section of the linear flow. We will use this perspective in subsection 3.2.

**Lemma 3.1.** Given any geometric armadillo tail with parameter  $r = \frac{1}{q}$ ,  $q \in \mathbb{N} \setminus \{1\}$ , there exists a saddle connection that we call  $bsc'_2$ .

*Proof.* We use the following notations:

- $\xrightarrow{x}$  indicates the linear flow where the horizontal displacement is *x*,
- $\stackrel{\text{portal}(\#)}{\longrightarrow}$  indicates that the flow hit a portal of  $\Box_{\#}$ , i.e., the *x*-coordinate is  $1 + \cdots + \frac{1}{q^{\#-1}}$  and the *y*-coordinate lies between  $\frac{1}{q^{\#}}$  and  $\frac{1}{q^{\#-1}}$ , hence it is identified to the corresponding point on the *y*-axis.
- $\xrightarrow{\text{roof}(\#)}$  indicates that the flow hits the roof of  $\Box_{\#}$  which is identified to the base of  $\Box_{\#}$ , hence we adjust the *y*-coordinate.

We start from  $\left(1+\frac{1}{q},0\right)$  and flow in the  $\frac{1}{2-r}$ -direction to construct bsc<sub>2</sub>.

$$\begin{pmatrix} 1+\frac{1}{q}, 0 \end{pmatrix} \xrightarrow{\frac{1}{q^2}} \begin{pmatrix} 1+\frac{1}{q}+\frac{1}{q^2}, \frac{1}{q(2q-1)} \end{pmatrix} \xrightarrow{\text{portal}(3)} \begin{pmatrix} 0, \frac{1}{q(2q-1)} \end{pmatrix} \\ \xrightarrow{1} \begin{pmatrix} 1, \frac{1+q^2}{q(2q-1)} \end{pmatrix} \xrightarrow{\text{portal}(1)} \begin{pmatrix} 0, \frac{1+q^2}{q(2q-1)} \end{pmatrix} \xrightarrow{\text{roof}(1)} \begin{pmatrix} 1, \frac{1+q}{q(2q-1)} \end{pmatrix}.$$

All operations above apply to all  $q \in \mathbb{N} \setminus \{1\}$ . Observe that  $\frac{1+q}{q(2q-1)} = \frac{1}{q}$  if q = 2, and  $\frac{1+q}{q(2q-1)} < \frac{1}{q}$  if q > 2. That is, if q = 2, the linear flow in the  $\frac{q}{2q-1}$ -direction from  $\left(1 + \frac{1}{q}, 0\right)$  to  $\left(1, \frac{1+q}{q(2q-1)}\right)$  is a saddle connection, bsc<sub>2</sub>'. If q > 2, we have  $\frac{1+q}{q(2q-1)} < \frac{1}{q}$ . Hence we do not yet have a saddle connection, and we enter  $\Box_2$ . Continuing from the last expression,

$$\begin{pmatrix} 1, \frac{1+q}{q(2q-1)} \end{pmatrix} \xrightarrow{\frac{1}{q}} \begin{pmatrix} 1+\frac{1}{q}, \frac{1+2q}{q(2q-1)} \end{pmatrix} \xrightarrow{\operatorname{roof}(2)} \begin{pmatrix} 1+\frac{1}{q}, \frac{2}{q(2q-1)} \end{pmatrix} \\ \xrightarrow{\text{portal}(2)} \begin{pmatrix} 0, \frac{2}{q(2q-1)} \end{pmatrix} \xrightarrow{\frac{1}{q}} \begin{pmatrix} 1, \frac{2+q^2}{q(2q-1)} \end{pmatrix} \xrightarrow{\text{portal}(1)} \begin{pmatrix} 0, \frac{2+q^2}{q(2q-1)} \end{pmatrix} \\ \xrightarrow{\frac{1}{q}} \begin{pmatrix} 1, \frac{2+2q^2}{q(2q-1)} \end{pmatrix} \xrightarrow{\operatorname{roof}(1)} \begin{pmatrix} 1, \frac{2+q}{q(2q-1)} \end{pmatrix}.$$

Note that  $\frac{2+q}{q(2q-1)} = \frac{1}{q}$  if q = 3, and  $\frac{2+q}{q(2q-1)} < \frac{1}{q}$  if q > 3. Hence we have a saddle connection  $bsc'_2$  for q = 3.

Induction hypothesis  $\frac{n+q}{q(2q-1)} = \frac{1}{q}$  if q = n + 1. Inductive step We show that  $\frac{n+1+q}{q(2q-1)} = \frac{1}{q}$  if q = n + 2.

Assume that the linear flow has not yielded a saddle connection yet, i.e., n > q - 1. Then,

$$\begin{pmatrix} 1, \frac{n+q}{q(2q-1)} \end{pmatrix} \xrightarrow{1/q} \begin{pmatrix} 1+\frac{1}{q}, \frac{n+2q}{q(2q-1)} \end{pmatrix} \xrightarrow{\text{roof}(1)} \begin{pmatrix} 1+\frac{1}{q}, \frac{n+1}{q(2q-1)} \end{pmatrix}$$

$$\xrightarrow{\text{portal}(2)} \begin{pmatrix} 0, \frac{n+1}{q(2q-1)} \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1, \frac{n+1+q^2}{q(2q-1)} \end{pmatrix} \xrightarrow{\text{portal}(1)} \begin{pmatrix} 0, \frac{n+1+q^2}{q(2q-1)} \end{pmatrix}$$

$$\xrightarrow{1} \begin{pmatrix} 1, \frac{n+1+2q^2}{q(2q-1)} \end{pmatrix} \xrightarrow{\text{roof}(1)} \begin{pmatrix} 1, \frac{n+1+q}{q(2q-1)} \end{pmatrix}$$

Since  $\frac{n+1+q}{q(2q-1)} = \frac{1}{q}$  for n+1 = q-1, this concludes our proof that  $bsc_2'$  exists for all  $q \in \mathbb{N} \setminus \{1\}$ .

In the following theorem, for every  $q \in \mathbb{N} \setminus \{1\}$ , we construct  $bsc'_k$  for every integer k > 2. The previous lemma serves as the base case for an induction proof that shows the existence of this saddle connection.

**Theorem 3.2.** Given a geometric armadillo tail with parameter  $r = \frac{1}{q}$ ,  $q \in \mathbb{N} \setminus \{1\}$ , there exists a saddle connection that we call  $bsc'_k$ .

*Proof.* We use the same notations we did in the previous lemma.

$$\begin{split} \left(1+\dots+\frac{1}{q^{k-1}},0\right) \xrightarrow{1/q^k} \left(1+\dots+\frac{1}{q^k},\frac{1}{q^{k-1}(2q-1)}\right) \\ & \underbrace{\underset{q^{k-1}(2q-1)}{\overset{portal(k+1)}{\overset{portal(k+1)}{\overset{portal(k+1)}{\overset{portal(k+1)}{\overset{portal(k+1)}{\overset{portal(2q-1)}{\overset$$

The last point above is a singularity if q = 2 and k = 2, however, we assume k > 2 and continue.

$$\cdots \xrightarrow{1/q} \left( 1 + \frac{1}{q}, \frac{1 + 2q^{k-1}}{q^{k-1}(2q-1)} \right) \xrightarrow{\text{roof}(2)} \left( 1 + \frac{1}{q}, \frac{1 + q^{k-2}}{q^{k-1}(2q-1)} \right).$$

Again, the last expression is a singularity if q = 2 and k = 3. **Induction hypothesis** For *l* small enough, assume

$$\left(1+\cdots+\frac{1}{q^{l}},\frac{1+q^{k-l-1}}{q^{k-1}(2q-1)}\right)=\left(1+\cdots+\frac{1}{q^{l}},\frac{1}{q^{l+1}}\right),$$

a singularity if q = 2.

Inductive step We show that

$$\left(1+\cdots+\frac{1}{q^{l+1}},\frac{1+q^{k-l-2}}{q^{k-1}(2q-1)}\right) = \left(1+\cdots+\frac{1}{q^{l+1}},\frac{1}{q^{l+2}}\right)$$

if q = 2.

For q > 2, we then have

$$\xrightarrow{1/q^{l+1}} \left( 1 + \dots + \frac{1}{q^{l+1}}, \frac{1 + 2q^{k-l-1}}{q^{k-1}(2q-1)} \right)$$

$$\xrightarrow{\text{roof}(l+2)}} \left( 1 + \dots + \frac{1}{q^{l+1}}, \frac{1 + q^{k-l-2}}{q^{k-1}(2q-1)} \right),$$

and the last expression is a singularity if q = 2, and this proves our hypothesis.

Let l = k - 3, then the last expression becomes  $\left(1 + \dots + \frac{1}{q^{k-2}}, \frac{1+q}{q^{k-1}(2q-1)}\right)$ , which is a singularity for q = 2. For q > 2, we have

$$\begin{pmatrix} 1 + \dots + \frac{1}{q^{k-2}}, \frac{1+q}{q^{k-1}(2q-1)} \end{pmatrix} \xrightarrow{1/q^{k-1}} \begin{pmatrix} 1 + \dots + \frac{1}{q^{k-1}}, \frac{1+2q}{q^{k-1}(2q-1)} \end{pmatrix} \\ \xrightarrow{\text{roof}(k)} \begin{pmatrix} 1 + \dots + \frac{1}{q^{k-1}}, \frac{2}{q^{k-1}(2q-1)} \end{pmatrix} \xrightarrow{\text{portal}(k)} \begin{pmatrix} 0, \frac{2}{q^{k-1}(2q-1)} \end{pmatrix} \\ \xrightarrow{1} \begin{pmatrix} 1, \frac{2+q^k}{q^{k-1}(2q-1)} \end{pmatrix} \xrightarrow{\text{portal}(1)} \begin{pmatrix} 1, \frac{2+2q^k}{q^{k-1}(2q-1)} \end{pmatrix} \xrightarrow{\text{roof}(1)} \begin{pmatrix} 1, \frac{2+q^{k-1}}{q^{k-1}(2q-1)} \end{pmatrix} \\ \xrightarrow{\text{portal}(1)} \xrightarrow{\text{portal}(1)} \begin{pmatrix} 1, \frac{2+2q^k}{q^{k-1}(2q-1)} \end{pmatrix} \xrightarrow{\text{roof}(1)} \begin{pmatrix} 1, \frac{2+q^{k-1}}{q^{k-1}(2q-1)} \end{pmatrix} \end{pmatrix}$$

which is a singularity if q = 3 and k = 2.

We claim that the rest follows from the induction technique used in Lemma 3.1.

3.2. **Constructing**  $bsc_k$ . To finish the construction of  $bsc_k$ , in lieu of tracking trajectories through portals and roofs, we will use an interval exchange transformation. If we take a section of the linear flow along the x = 0 in the polygonal representation, we see an infinite interval exchange transformation. However, as we will see, the saddle connections in  $bsc_k$  never extend beyond *k*-squares, which means we will only need to use what is essentially a 2k-IET.

To sample how this works, we will construct 
$$bsc_3$$
 using the map  $T_3 : [0,1) \setminus U \rightarrow [0,1) \setminus V$ ,  
where  $U = \left(\frac{q^3 - 1}{q^2(2q - 1)}, \frac{q^4 + q - 1}{q^3(2q - 1)}\right)$  and  $V = \left(0, \frac{1}{q^3}\right)$ :

$$T_{3}(x) = \begin{cases} x + \frac{q}{2q - 1}, & x \in \left[0, \frac{q - 1}{2q - 1}\right) \\ 0, & x = \frac{q - 1}{2q - 1} \\ x + \frac{2 - q}{2q - 1}, & x \in \left(\frac{q - 1}{2q - 1}, \frac{q^{2} - 1}{q(2q - 1)}\right] \\ x + \frac{2 - q^{2}}{q(2q - 1)}, & x \in \left(\frac{q^{2} - 1}{q(2q - 1)}, \frac{q^{3} - 1}{q^{2}(2q - 1)}\right] \\ x + \frac{1 - q^{3}}{q^{2}(2q - 1)}, & x \in \left[\frac{q^{4} + q - 1}{q^{3}(2q - 1)}, \frac{q^{3} + q - 1}{q^{2}(2q - 1)}\right] \\ x + \frac{1 - q^{2}}{q(2q - 1)}, & x \in \left[\frac{q^{3} + q - 1}{q^{2}(2q - 1)}, \frac{q^{2} + q - 1}{q(2q - 1)}\right] \\ x + \frac{1 - q}{2q - 1}, & x \in \left[\frac{q^{2} + q - 1}{q(2q - 1)}, 1\right]. \end{cases}$$

Both  $bsc_3$  and  $T_3$  depend on the parameter q. In Figure 6, we give a visual description of the map  $T_3$ . Note that the black region corresponds to where  $T_3$  is left undefined.



FIGURE 6. 6-IET on  $X_3$ 

**Lemma 3.3.** Define  $bsc_3 = bsc_2 + \frac{q-1}{q^2(2q-1)} \oplus bsc'_3$ . Then  $bsc_3$  is a concatenation of two saddle connections.

*Proof.* Clearly,  $bsc'_3$  is one of the two saddle connections. We show that  $bsc_2 + \frac{q-1}{q^2(2q-1)}$  yields a single saddle connection.

Observe that  $bsc'_2$  is one of the two saddle connections in  $bsc_2$ . We can shift  $bsc'_2$  vertically by  $\frac{q-1}{q^2(2q-1)}$ . When we take the closure of the vertical shift, we see endpoints corresponding to  $\left(1 + \frac{1}{q}, \frac{q-1}{q^2(2q-1)}\right)$  and  $\left(1, \frac{1}{q} + \frac{q-1}{q^2(2q-1)}\right) = \left(1, \frac{2q^2-1}{q^2(2q-1)}\right)$ , neither of which are singular points. (See the construction of  $bsc'_2$  in Lemma 3.1.) The segment does not cross over the singular point as we continuously push the segment vertically, however, we see that the segment intersects that singular point at exactly one point:  $\left(1 + \frac{1}{q} + \frac{1}{q^2}, \frac{1}{q(2q-1)} + \frac{q-1}{q^2(2q-1)}\right) = \left(1 + \frac{1}{q} + \frac{1}{q^2}, \frac{1}{q^2}\right)$ . Further note that  $\left(1, \frac{2q^2-1}{q^2(2q-1)}\right)$  is identified to a point on the section x = 0 since  $\left(1, \frac{2q^2-1}{q^2(2q-1)}\right) \sim \frac{1}{q}$ .

Further note that  $\left(1, \frac{2q^2-1}{q^2(2q-1)}\right)$  is identified to a point on the section x = 0 since  $\left(1, \frac{2q^2-1}{q^2(2q-1)}\right) \sim \left(0, \frac{2q^2-1}{q^2(2q-1)}\right)$ . Similarly, the singular point  $\left(1 + \frac{1}{q} + \frac{1}{q^2}, \frac{1}{q^2}\right) \sim \left(0, \frac{1}{q^2}\right)$ . We will show that the linear

flow connects  $\left(0, \frac{2q^2-1}{q^2(2q-1)}\right)$  to  $\left(0, \frac{1}{q^2}\right)$  using  $T_3$ , and in doing so, we will see that all iterates of the map avoid the singular point.

Observe that when q = 2,  $T_3\left(\frac{2q^2-1}{q^2(2q-1)}\right) = \frac{1}{q^2}$ , and we are done. For general q, we make the following observation.

**Claim 3.4.** For any  $q \in \mathbb{N} \setminus \{0\}$ , any  $2 \le n \le q$ ,

$$T_3^{2n-4}\left(\frac{2q^2-1}{q^2(2q-1)}\right) = \frac{nq^2-1}{q^2(2q-1)}$$

The proof of the claim follows from induction, where q = 2 is our base case. Indeed,

$$T_3^{2n-4}\left(\frac{2q^2-1}{q^2(2q-1)}\right) = T_3^2 \circ T_3^{2(n-1)-4}\left(\frac{2q^2-1}{q^2(2q-1)}\right) = T_3^2\left(\frac{(n-1)q^2-1}{q^2(2q-1)}\right)$$

by the inductive hypothesis. Observe that for any  $n \leq q$ ,

$$\frac{(n-1)q^2-1}{q^2(2q-1)} \in \left[0, \frac{q-1}{2q-1}\right)$$

since  $(n-1)q^2 - 1 < q^3 - q^2$  for any  $n \le q$ . Consequently,

$$T_3^2\left(\frac{(n-1)q^2-1}{q^2(2q-1)}\right) = T_3\left(\frac{(n-1)q^2-1}{q^2(2q-1)} + \frac{q}{2q-1}\right) = T_3\left(\frac{q^3+(n-1)q^2-1}{q^2(2q-1)}\right).$$

Now, observe that for any  $2 \le n \le q$ ,

$$\frac{q^3 + (n-1)q^2 - 1}{q^2(2q-1)} \in \left[\frac{q^2 + q - 1}{q(2q-1)}, 1\right)$$

Indeed,

$$\frac{q^2 + q - 1}{q(2q - 1)} \le \frac{q^3 + (n - 1)q^2 - 1}{q^2(2q - 1)} < 1$$

since, first,  $1 \le (n-2)q^2 + q$  for any  $n \ge 2$ , and second,  $nq^2 < 2q^3 + 1$  for  $n \le q$ . Consequently,

$$T_3\left(\frac{q^3 + (n-1)q^2 - 1}{q^2(2q-1)}\right) = \frac{q^3 + (n-1)q^2 - 1}{q^2(2q-1)} + \frac{1-q}{2q-1} = \frac{kq^2 - 1}{q^2(2q-1)},$$

as desired. This completes the proof of the claim.

To complete the proof of the Lemma, take n = q and we have

$$T_3^{2q-4}\left(\frac{2q^2-1}{q^2(2q-1)}\right) = \frac{q^3-1}{q^2(2q-1)}$$

and since 
$$\frac{q^3 - 1}{q^2(2q - 1)} \in \left(\frac{q^2 - 1}{q(2q - 1)}, \frac{q^3 - 1}{q^2(2q - 1)}\right],$$
  
 $T_3\left(\frac{q^3 - 1}{q^2(2q - 1)}\right) = \frac{q^3 - 1}{q^2(2q - 1)} + \frac{2 - q^2}{q(2q - 1)} = \frac{1}{q^2},$ 

which completes the proof.

An analogous proof works for  $\operatorname{bsc}_k$ . However, we will need to use the map  $T_k : [0,1) \setminus U \to [0,1) \setminus V$ , where  $U = \left(\frac{q^k - 1}{q^{k-1}(2q-1)}, \frac{q^{k+1} + q - 1}{q^k(2q-1)}\right)$  and  $V = \left(0, \frac{1}{q^k}\right)$  defined by:

$$\begin{aligned} & \left\{ \begin{aligned} x + \frac{q}{2q-1}, & x \in \left[0, \frac{q-1}{2q-1}\right) \\ 0, & x = \frac{q-1}{2q-1} \\ x + \frac{2-q}{2q-1}, & x \in \left(\frac{q-1}{2q-1}, \frac{q^2-1}{q(2q-1)}\right] \\ x + \frac{2-q^2}{q(2q-1)}, & x \in \left(\frac{q^{2}-1}{q(2q-1)}, \frac{q^3-1}{q^2(2q-1)}\right] \\ \vdots \\ x + \frac{2-q^{k-i+1}}{q^{k-i}(2q-1)}, & x \in \left(\frac{q^{k-i+1}-1}{q^{k-i}(2q-1)}, \frac{q^{k-i+2}-1}{q^{k-i+1}(2q-1)}\right) \\ \vdots \\ x + \frac{2-q^{k-i}}{q^{k-2}(2q-1)}, & x \in \left(\frac{q^{k-1}-1}{q^{k-2}(2q-1)}, \frac{q^{k}-1}{q^{k-1}(2q-1)}\right) \\ x + \frac{1-q^k}{q^{k-1}(2q-1)}, & x \in \left[\frac{q^{k+1}+q-1}{q^{k}(2q-1)}, \frac{q^k+q-1}{q^{k-1}(2q-1)}\right) \\ \vdots \\ x + \frac{1-q^{k}}{q^{k-i}(2q-1)}, & x \in \left[\frac{q^{k-j+2}+q-1}{q^{k-j+1}(2q-1)}, \frac{q^{k-j+1}+q-1}{q^{k-i}(2q-1)}\right) \\ \vdots \\ x + \frac{1-q^3}{q^2(2q-1)}, & x \in \left[\frac{q^4+q-1}{q^3(2q-1)}, \frac{q^3+q-1}{q^2(2q-1)}\right) \\ x + \frac{1-q^2}{q(2q-1)}, & x \in \left[\frac{q^3+q-1}{q^2(2q-1)}, \frac{q^2+q-1}{q(2q-1)}\right) \\ x + \frac{1-q}{2q-1}, & x \in \left[\frac{q^2+q-1}{q(2q-1)}, 1\right), \end{aligned} \end{aligned}$$

where  $i, j \in \mathbb{N}$ ,  $0 < j \le k$  and  $1 < i \le k$ . As with  $T_3$  and bsc<sub>3</sub>, both  $T_k$  and bsc<sub>k</sub> depend on the parameter *q*.

**Theorem 3.5.** For k > 2, define  $bsc_k = bsc_{k-1} + \frac{q-1}{q^{k-1}(2q-1)} \oplus bsc'_k$ . Then  $bsc_k$  is a concatenation of two saddle connections.

*Proof.* Clearly,  $bsc'_k$  is one of the two saddle connections. We show that  $bsc_{k-1} + \frac{q-1}{q^{k-1}(2q-1)}$  yields a single saddle connection.

Analogous to the previous lemma, observe that  $bsc'_{k-1}$  is one of the two saddle connections in  $bsc_{k-1}$ . We can shift  $bsc'_{k-1}$  vertically by  $\frac{q-1}{q^{k-1}(2q-1)}$ . When we take the closure of the vertical shift, we see endpoints corresponding to

$$\left(1+\frac{1}{q}+\cdots+\frac{1}{q^{k-1}},\frac{q-1}{q^{k-1}(2q-1)}\right)$$

and

$$\left(1+\frac{1}{q}+\cdot+\frac{1}{q^{k-2}},\frac{1}{q^{k-1}}+\frac{q-1}{q^{k-1}(2q-1)}\right) = \left(1+\frac{1}{q}+\cdot+\frac{1}{q^{k-2}},\frac{2q^2-1}{q^{k-1}(2q-1)}\right)$$

neither of which are singular points. (See the construction of  $bsc'_k$  in Theorem 3.2.) The segment does not cross over the singular point as we continuously push the segment vertically, however,

we see that the segment intersects that singular point at exactly one point:

$$\left(1+\frac{1}{q}+\cdots+\frac{1}{q^{k-1}},\frac{1}{q^{k-2}(2q-1)}+\frac{q-1}{q^{k-1}(2q-1)}\right)=\left(1+\frac{1}{q}+\cdots+\frac{1}{q^{k-1}},\frac{1}{q^{k-1}}\right).$$

Further note that  $\left(1 + \frac{1}{q} + \cdot + \frac{1}{q^{k-2}}, \frac{2q^2-1}{q^{k-1}(2q-1)}\right)$  is identified to a point on the section x = 0 since

$$\left(1+\frac{1}{q}+\cdot+\frac{1}{q^{k-2}},\frac{2q^2-1}{q^{k-1}(2q-1)}\right)\sim\left(0,\frac{2q^2-1}{q^{k-1}(2q-1)}\right)$$

Similarly, the singular point  $\left(1 + \frac{1}{q} + \dots + \frac{1}{q^{k-1}}, \frac{1}{q^{k-1}}\right) \sim \left(0, \frac{1}{q^{k-1}}\right)$ . We will show that the linear flow connects  $\left(0, \frac{2q^2-1}{q^{k-1}(2q-1)}\right)$  to  $\left(0, \frac{1}{q^{k-1}}\right)$  using  $T_k$ , and in doing so, we will see that all iterates of the map avoid the singular point.

**Claim 3.6.** For any  $q \in \mathbb{N} \setminus \{0\}$ , any  $2 \le n \le q$ , and any  $j \le k$ ,

$$T_k^{2n-4}\left(\frac{2q^{j-1}-1}{q^{k-1}(2q-1)}\right) = \frac{nq^{j-1}-1}{q^{k-1}(2q-1)}$$

The proof of the claim follows from induction, where q = 2 is our (trivial) base case. Indeed,

$$T_k^{2n-4}\left(\frac{2q^{j-1}-1}{q^{k-1}(2q-1)}\right) = T_k^2 \circ T_k^{2(n-1)-4}\left(\frac{2q^{j-1}-1}{q^{k-1}(2q-1)}\right) = T_k^2\left(\frac{(n-1)q^{j-1}-1}{q^{k-1}(2q-1)}\right)$$

by the inductive hypothesis. Observe that for any  $j \le k$  and  $n \le q$ ,

$$\frac{(n-1)q^{j-1}-1}{q^{k-1}(2q-1)} \in \left[0, \frac{q-1}{2q-1}\right)$$

since  $(n-1)q^{j-1} - 1 < q^k - q^{k-1}$  for any  $j \le k$  and  $n \le q$ . Consequently,

$$T_k^2\left(\frac{(n-1)q^{j-1}-1}{q^{k-1}(2q-1)}\right) = T_k\left(\frac{(n-1)q^{j-1}-1}{q^{k-1}(2q-1)} + \frac{q}{2q-1}\right) = T_k\left(\frac{q^k + (n-1)q^{j-1}-1}{q^{k-1}(2q-1)}\right).$$

Now, observe that for any  $2 \le n \le q$ ,

$$\frac{q^{k} + (n-1)q^{j-1} - 1}{q^{k-1}(2q-1)} \in \left[\frac{q^{k-j+2} + q - 1}{q^{k-j+1}(2q-1)}, \frac{q^{k-j+1} + q - 1}{q^{k-j}(2q-1)}\right)$$

Indeed,

$$\frac{q^{k-j+2}+q-1}{q^{k-j+1}(2q-1)} \le \frac{q^k+(n-1)q^{j-1}-1}{q^{k-1}(2q-1)} < \frac{q^{k-j+1}+q-1}{q^{k-j}(2q-1)}$$

since, first,  $q^k + q^{j-1} - 1 \le q^k + (n-1)q^{j-1} - 1$  for any  $n \ge 2$ , and second,

$$q^{k} + (n-1)q^{j-1} - 1 < q^{k} + q^{j} - 1$$

for  $n \leq q$ .

Consequently,

$$T_k\left(\frac{q^k + (n-1)q^{j-1} - 1}{q^{k-1}(2q-1)}\right) = \frac{q^k + (n-1)q^{j-1} - 1}{q^{k-1}(2q-1)} + \frac{1 - q^{k-j+1}}{q^{k-j}(2q-1)} = \frac{mq^{j-1} - 1}{q^{k-1}(2q-1)},$$

as desired. This completes the proof of the claim.

Now, take n = q and we have

$$T_k^{2q-4}\left(\frac{2q^2-1}{q^{k-1}(2q-1)}\right) = \frac{q^3-1}{q^{k-1}(2q-1)}$$

Observe that  $\frac{q^{j}-1}{q^{k-1}(2q-1)} \in \left[0, \frac{q-1}{2q-1}\right)$  since  $q^{j}-1 < q^{k}-1$  for any  $j \le k$ . Apply  $T_{k}$  again, and we see  $T_{k}\left(\frac{q^{j-1}-1}{1+1}\right) = \frac{q^{j-1}-1}{1+1} + \frac{q}{1+1} = \frac{q^{k}+q^{j}-1}{1+1}$ .

$$T_k\left(\frac{1}{q^{k-1}(2q-1)}\right) = \frac{1}{q^{k-1}(2q-1)} + \frac{1}{2q-1} = \frac{1}{q^{k-1}(2q-1)}.$$

Iterating again, we have that

$$\frac{q^{k}+q^{j}-1}{q^{k-1}(2q-1)} \in \left[\frac{q^{k-j+1}+q-1}{q^{k-j}(2q-1)}, \frac{q^{k-j}+q-1}{q^{k-j-1}(2q-1)}\right)$$

since  $\frac{q^k + q^j - 1}{q^{k-1}(2q-1)} = \frac{q^{k-j+1} + q - 1}{q^{k-j}(2q-1)}$  and consequently  $T_k \left( \frac{q^k + q^j - 1}{q^{k-1}(2q-1)} \right) = \frac{q^k + q^j - 1}{q^{k-1}(2q-1)} + \frac{1 - q^{k-j}}{q^{k-j-1}(2q-1)} = \frac{2q^j - 1}{q^{k-1}(2q-1)}.$ 

This provides a road map which will complete the proof. We have:

$$\begin{split} T_k^{2q-2} \left( \frac{2q^{j-1}-1}{q^{k-1}(2q-1)} \right) &= T_k^2 \circ T_k^{2q-4} \left( \frac{2q^{j-1}-1}{q^{k-1}(2q-1)} \right) \\ &= T_k^2 \left( \frac{q^j-1}{q^{k-1}(2q-1)} \right) \\ &= \frac{2q^j-1}{q^{k-1}(2q-1)}. \end{split}$$

Consequently,

$$T_k^{(k-3)(2q-2)}\left(\frac{2q^2-1}{q^{k-1}(2q-1)}\right) = \frac{2q^{k-1}-1}{q^{k-1}(2q-1)}$$

and

$$T_k^{2q-4}\left(\frac{2q^{k-1}-1}{q^{k-1}(2q-1)}\right) = \frac{q^k-1}{q^{k-1}(2q-1)}.$$

We apply  $T_k$  one last time, observing that

$$\frac{q^{k}-1}{q^{k-1}(2q-1)} \in \left(\frac{q^{k-1}-1}{q^{k-2}(2q-1)}, \frac{q^{k}-1}{q^{k-1}(2q-1)}\right],$$

and we see

$$T_k\left(\frac{q^k-1}{q^{k-1}(2q-1)}\right) = \frac{q^k-1}{q^{k-1}(2q-1)} + \frac{2-q^{k-1}}{q^{k-2}(2q-1)} = \frac{1}{q^{k-1}},$$

as desired. In summary,

$$T_k^{(k-3)(2q-2)+2q-3}\left(\frac{2q^2-1}{q^{k-1}(2q-1)}\right) = \frac{1}{q^{k-1}}.$$

We record the intersections of  $bsc_k$  with the section x = 0 for arbitrary k, using the construction of the saddle connections above.

**Theorem 3.7.** The saddle connections  $bsc_k$  intersects  $\{([0, y) : 0 < y < 1\}$  at points where *y* is an element of

$$\begin{cases} \frac{1}{2q-1}, \cdots, \frac{\widehat{q}}{2q-1}, \cdots, \frac{2q-2}{2q-1} \end{cases} + \sum_{i=1}^{k-1} \frac{q-1}{q^i(2q-1)} \\ \cup \quad \left\{ \frac{i}{q(2q-1)}, \frac{i+q^2}{q(2q-1)} \right\}_{i=1}^{q-1} + \sum_{i=2}^{k-1} \frac{q-1}{q^i(2q-1)} \\ \vdots \\ \cup \quad \left\{ \frac{i}{q^{j-1}(2q-1)}, \frac{i+q^j}{q^{j-1}(2q-1)} \right\}_{i=1}^{q-1} + \sum_{i=j}^{k-1} \frac{q-1}{q^i(2q-1)} \\ \vdots \\ \cup \quad \left\{ \frac{i}{q^{k-1}(2q-1)}, \frac{i+q^k}{q^{k-1}(2q-1)} \right\}_{i=1}^{q-1} \end{cases}$$

where "set + number" is a set where the number is added to every element in the set.

*Proof.* The saddle connections  $bsc_k$  consists of vertical shifts of each  $bsc'_j$  for every 1 < j < k (where the vertical shift depends on *j*), as well as  $bsc'_k$  and a vertical shift of  $bsc_1$ . Thus, the intersections of  $bsc_k$  correspond to the intersections of  $bsc_1$  shifted up by  $\sum_{i=1}^{k-1} \frac{q-1}{q^i(2q-1)}$ ,  $bsc'_2$  shifted up by  $\sum_{i=2}^{k-1} \frac{q-1}{q^i(2q-1)}$ , and so on, until we reach  $bsc'_k$ .

## 4. INFINITELY MANY CYLINDERS (OR FINDING SUCCESSIVE CYLINDERS IN THE $\frac{1}{2-r}$ -DIRECTION)

In this section, we define a map  $f_r$  to construct new cylinders from existing cylinders.

**Definition 4.1.** The map  $\tilde{f}_r : \mathbb{R}^2 \to \mathbb{R}^2$ , where

$$\tilde{f}_r: \begin{pmatrix} x\\ y \end{pmatrix} \mapsto \begin{pmatrix} rx+1\\ ry \end{pmatrix}.$$

Observe that  $\tilde{f}_r$  is an injective map.

The map does not descend from  $\mathbb{R}^2$  to a well-defined map on the quotient of the polygonal representation of an armadillo tail. The issue is that the map does not respect the vertical gluings (by horizontal translations). For instance, the left edge of  $\Box_1$  is mapped to the left edge of  $\Box_2$ , but the left edge of  $\Box_2$  is glued to the *right* edge of  $\Box_1$ . However, the map does descend to a partial quotient, where we only identify the top and bottom edges, since  $\tilde{f}_r$  respects the identifications along the tops and bottoms of the squares. That is the content of the following lemma, whose proof is elementary.

**Lemma 4.2.** Let  $P_r$  be the polygonal representation of an armadillo tail *X* with parameter *r* such that the polygonal representation is embedded in  $\mathbb{R}^2$  and the edge identifications forgotten. Let  $X^{tb}$ 

be a quotient of  $P_r$  by identifying the top and bottom edges only. Let  $\Box_k^{tb}$  denote the *k*th-square in  $X^{tb}$ . Then  $\tilde{f}_r$  descends to a map  $f_r$  on  $X^{tb}$ . The image of  $\Box_k^{tb}$  under  $f_r$  is  $\Box_{k+1}^{tb}$ .

Let  $q : P_r \to X^{tb}$  be the quotient map identifying the top and bottom edges of the polygon. Let  $\operatorname{cyl}_k$  be a cylinder, and lift  $\operatorname{cyl}_k$  to  $X^{tb}$ . Call this lift  $\operatorname{cyl}_k^{tb}$ . Let *L* and *R* denote the unidentified left and right edges of the polygon. Inductively define  $\operatorname{cyl}_{k+1}$  as the closure of  $q \circ f_r(\operatorname{cyl}_k^{tb} \setminus (L \cup R))$  with respect to the linear flow in the  $\frac{1}{2-r}$ -direction. Observe that  $\operatorname{cyl}_{k+1}$  does not depend on the chosen lift of  $\operatorname{cyl}_k$ .

Given a geometric armadillo tail with parameter r = 1/q, first we show that  $cyl_k$  lies entirely on  $X_{k+1} = \bigcup_{i=1}^{k+1} \Box_i$ . Then  $q \circ f_r(cyl_k)$  is a subset of a cylinder that lies in  $X_{k+1}^{tb} \setminus \Box_1$ . We call this a *partial cylinder*. We will show that there is a circle rotation on  $\{0\} \times [0, 1]$  that fills in  $q \circ f_r(cyl_k)$  at the points of discontinuity.

We define where the circle rotation is defined, and prove that waist curves of cylinders are periodic points under the circle rotation.

We define the *generation zone* as the preimage of  $cyl_1$  in  $X \setminus \Box_1$  under  $f_r$ :

generation zone := 
$$f_r^{-1}(\text{Int}(\text{cyl}_1 \cap \Box_2))$$
  
=  $\left\{ (x, y) : \frac{1}{2-r}x < y < \frac{1}{2-r}x + \frac{1-r}{2-r}, 0 \le x \le 1 \right\}.$ 

Given the set of points where  $\text{cyl}_k$  intersects  $\{0\} \times [0, 1]$  and  $\{1\} \times [0, 1]$ , we remove the points that lie in the generation zone. Define sets  $S_1$  (and  $S_2$ , resp.) on  $\{0\} \times [0, 1]$  (and  $\{1\} \times [0, 1]$ , resp.) as the image of the remaining points under  $f_r$ . That is,

$$S_1 = \operatorname{proj}_y \circ f_r\left(\gamma \cap \left\{(0, y) : 0 < y < \frac{1}{2 - r}\right\}\right)$$

and

$$S_2 = \left\{ f_r \left( \gamma \cap \left\{ (0, y) : \frac{1 - r}{2 - r} < y < 1 \right\} \right) \right\}$$

where  $\text{proj}_{y}(x, y) = (0, y)$  is the projection onto the *y*-axis.

Recall the circle rotation  $T(x) = x + \frac{q}{2q-1}$ . Note that  $q \circ f_r(\text{cyl}_k) \subset X^{tb} \setminus \Box_1$  is a subset of a cylinder. Since the circle rotation is a section of the linear flow, we "fill in"  $q \circ f_r(\text{cyl}_k)$  at the points of discontinuity to construct  $\text{cyl}_{k+1}$ . Figure 7 illustrates the setting.



FIGURE 7.  $cyl_1 \setminus generation zone(left)$ ,  $f_r(cyl_1 \setminus generation zone)$  (center), connecting  $S_1$  and  $S_2$  via the circle rotation T (right). The dotted lines represent  $bsc_2$  in  $\Box_1$ .

We illustrate the simplest case (k = 1) before we prove the general case for all k. Since we can explicitly write the points at which bsc<sub>1</sub> intersects the *y*-axis (Theorem 2.3) we list them here and

take a waist curve  $\gamma$  to be  $\epsilon$  above these points where  $0\epsilon < \frac{q-1}{q(2q-1)}$ . The *y*-coordinate of these points are

$$\left\{\frac{1}{2q-1},\ldots,\frac{q}{2q-1},\ldots,\frac{2q-2}{2q-1}\right\}$$

hence we have

$$S_1 = \left\{ (0,y) : y = \frac{1}{q(2q-1)}, \dots, \frac{q-1}{q(2q-1)} \right\}, \quad S_2 = \left\{ (1,y) : y = \frac{q-1}{q(2q-1)}, \dots, \frac{2q-2}{q(2q-1)} \right\}.$$

Take  $i = 1, \ldots, q - 2$ , then

$$\frac{i}{q(2q-1)} \xrightarrow{T} \frac{i+q^2}{q(2q-1)} \xrightarrow{T} \frac{i+2q^2}{q(2q-1)} \equiv \frac{i}{q(2q-1)},$$

where  $\frac{1}{q} < \frac{i+q^2}{q(2q-1)} < 1$  for  $i \in \{1, \dots, q-2\}$ . We note that  $\left\{\left(0, \frac{i+q^2}{q(2q-1)}\right)\right\}_{i=1}^{q-2}$  are q-2 additional points where we hit the *y*-axis.

When i = q - 1,

$$\frac{q-1}{q(2q-1)} \xrightarrow{T} \frac{q-1+q^2}{q(2q-1)} \xrightarrow{T} \frac{q-1+2q^2}{q(2q-1)} \equiv \frac{2q-1}{q(2q-1)}$$

First note that  $\left(1, \frac{2q-1}{q(2q-1)}\right)$  is a singularity. Since  $\gamma$  is slightly above this, we continue by iterating *T*. Moreover, note that  $T^{2q-1}\left(\frac{q-1}{q(2q-1)}\right) = \frac{q-1+(2q-1)q^2}{q(2q-1)} \equiv \frac{q-1}{q(2q-1)}$ . We will show below that for any m < 2q - 1, the *m*th iterate hits the portal of  $\Box_1$ , i.e.,  $\frac{1}{q} < T^m\left(\frac{q-1}{q(2q-1)}\right) = \frac{q-1+mq^2}{q(2q-1)} < 1$ . This adds an additional q - 2 points where we hit the *y*-axis.

**Definition 4.3.** Given  $cyl_k$ , we define the *skew-width of cyl\_k* as the vertical distance (vertical in relation to the polygonal representation) between the two boundary saddle connections of  $cyl_k$  through the cylinder.

**Lemma 4.4.** Given a geometric armadillo tail with parameter 1/q, the skew-width of  $\text{cyl}_k$  is  $\frac{q-1}{q^k(2q-1)}$ , and the width of  $\text{cyl}_k$  is  $\frac{q-1}{q^k\sqrt{q^2+(2q-1)^2}}$ .

**Theorem 4.5.** The circle rotation  $T : [0,1]/_{\sim} \rightarrow [0,1]/_{\sim}$  where  $T(x) = x + \frac{q}{2q-1}$  maps  $S_1$  to  $S_2$  defined above.

*Proof.* Theorem 3.7 tells us exactly where  $bsc_k$  intersects  $\{[0, y] : 0 < y < 1\}$ . Recall that  $S_1$  consists of points on  $\{0\} \times [0, 1]$  whose *y*-coordinates are

$$\frac{1}{q} \left( \bigcup_{j=1}^{k-1} \left\{ \frac{i}{q^{j-1}(2q-1)} \right\}_{i=1}^{q-1} + \sum_{i=j}^{k-1} \frac{q-1}{q^i(2q-1)} \cup \left\{ \frac{i}{q^{k-1}(2q-1)} \right\}_{i=1}^{q-1} \right)$$

and  $S_2$  consists of points on  $\{1\} \times [0, 1]$  whose *y*-coordinates are

$$\begin{array}{c} \frac{1}{q} \left( \left\{ \frac{q-1}{2q-1}, \frac{q+1}{2q-1}, \cdots, \frac{2q-2}{2q-1} \right\} + \sum_{i=1}^{k-1} \frac{q-1}{q^i(2q-1)} \right) \\ \cup \quad \bigcup_{j=2}^{k-1} \frac{1}{q} \left( \left\{ \frac{i+q^j}{q^{j-1}(2q-1)} \right\}_{i=1}^{q-1} + \sum_{i=j}^{k-1} \frac{q-1}{q^i(2q-1)} \right) \\ \cup \quad \frac{1}{q} \left\{ \frac{i+q^k}{q^{k-1}(2q-1)} \right\}_{i=1}^{q-1}. \end{array}$$

These can be simplified to

$$S_1 = \left\{ (0, y) : y \in \bigcup_{j=1}^{k-1} \left\{ \frac{(i+1)q^{k-j} - 1}{q^k(2q-1)} \right\}_{i=1}^{q-1} \cup \left\{ \frac{i}{q^k(2q-1)} \right\}_{i=1}^{q-1} \right\}$$

and

$$S_{2} = \left\{ (1,y) : y \in \left\{ \frac{(1+i)q^{k-1} + q^{k} - 1}{q^{k}(2q-1)} \right\}_{i=-1, i \neq 0}^{q-2} \cup \bigcup_{j=2}^{k-1} \left\{ \frac{(1+i)q^{k-j} + q^{k} - 1}{q^{k}(2q-1)} \right\}_{i=1}^{q-1} \cup \left\{ \frac{i+q^{k}}{q^{k-1}(2q-1)} \right\}_{i=1}^{q-1} \right\}$$

We will show that under some iterate of T,  $S_1$  maps onto  $S_2$ . We will break this down into three cases. In each case, we show that a point in  $S_1$  maps to a point in  $S_2$  under  $T^2$  (or  $T^{2q-1}$ ) but any

fewer iterate of *T* maps it to the complement of  $S_2$  on  $\{1\} \times [0,1]$ , i.e.,  $\left\{(1,y) : \frac{q}{2q-1} < y < 1\right\}$ . **Case 1.** First we have  $T\left(\frac{i}{q^k(2q-1)}\right) = \frac{i+q^k}{q^k(2q-1)}$  for any  $i \in \{1, \dots, q-1\}$ . **Case 2-1.** Consider the cases where  $j = 2, \dots, k-1$ , and  $i = 1, \dots, q-1$ . Then

$$T^{2}\left(\frac{(i+1)q^{k-j}-1}{q^{k}(2q-1)}\right) = \frac{(i+1)q^{k-j}-1+2q^{k+1}}{q^{k}(2q-1)} \equiv \frac{(i+1)q^{k-j}-1+q^{k}}{q^{k}(2q-1)}.$$

We show that  $T\left(\frac{(i+1)q^{k-j}-1}{q^k(2q-1)}\right)$  does not hit any point in  $S_2$ , i.e.,

$$\frac{1}{q} < \frac{(i+1)q^{k-j}-1+q^{k+1}}{q^k(2q-1)} < 1$$

The left inequality holds since it is equivalent to the inequalities below:

$$\begin{array}{ll} q^{k-1}(2q-1) &< (i+1)q^{k-j}-1+q^{k+1} \\ 1 &< (i+1)q^{k-j}+q^{k+1}-q^{k-1}(2q-1) \\ &= q^{k-1}\left((i+1)q^{1-j}+(q-1)^2\right) \end{array} ,$$

and the right inequality holds since it is equivalent to

$$q^k(q-1) > (i+q)q^{k-j}-1 \ q^k\left(q-1-(i+1)q^{-j}
ight)+1 \ge q^k\left(q-1-rac{q}{q^j}
ight)+1>0.$$

**Case 2-2.** Next, the cases where j = 1 and  $i = 1, \ldots, q - 2$  can be shown with the same technique as in Case 2-1: we have

$$T^{2}\left(\frac{(i+1)q^{k-1}-1}{q^{k}(2q-1)}\right) = \frac{(i+1)q^{k-1}-1+2q^{k}}{q^{k}(2q-1)} \equiv \frac{(i+1)q^{k-1}-1+q^{k}}{q^{k}(2q-1)}$$

and  $\frac{1}{q} < T\left(\frac{(i+1)q^{k-1}-1}{q^k(2q-1)}\right) < 1$ . The left inequality holds since

$$\begin{array}{rl} q^{k-1}(2q-1) &< (i+1)q^{k-1}-1+q^{k+1} \\ & 1 &< q^{k-1}\left(i+2+2q+q^2\right). \end{array}$$

However, the right inequality holds only for i = 1, ..., q - 2:

$$\begin{aligned} &(i+1)q^{k-1}-1+q^{k+1} &< 2q^{k+1}-q^k \\ &-1 &< q^{k+1}-(i+2)q^k = q^k\left(q-(i+2)\right). \end{aligned}$$

**Case 3.** Lastly, we deal with j = 1 and i = q - 1.

After 2q - 1-iterates,  $\frac{q^k - 1}{q^k(2q - 1)}$  is mapped to itself. We need to show that for any m < 2q - 1,  $T^m\left(\frac{q^k - 1}{q^k(2q - 1)}\right)$  falls between  $\frac{1}{q}$  and 1, hence does not hit any other point in  $S_2$ . If m = 2l, (l = 1, ..., q - 1), then

$$T^{m}\left(\frac{q^{k}-1}{q^{k}(2q-1)}\right) = \frac{q^{k}-1+2lq^{k}}{q^{k}(2q-1)} \equiv \frac{(l+1)q^{k}-1}{q^{k}(2q-1)}.$$

We show that

$$\frac{1}{q} < \frac{(l+1)q^k - 1}{q^k(2q - 1)} < 1.$$

The left-hand inequality is equivalent to

$$\begin{array}{ll} q^{k-1}(2q-1) &< (l+1)q^k-1 \\ 1 &< (l+1)q^k-q^{k-1}(2q-1) = q^{k-1}\left((l+1)q-(2q-1)\right) = \left((l-1)q+1\right), \end{array}$$

and the right-hand inequality is equivalent to

$$(l+1)q^k - 1 < q^k(2q-1) \\ -1 < q^k(2q-2-l).$$

If m = 2l + 1, (l = 1, ..., q - 2), then

$$T^{m}\left(\frac{q^{k}-1}{q^{k}(2q-1)}\right) = \frac{q^{k}-1+(2l+1)q^{k+1}}{q^{k}(2q-1)} \equiv \frac{(l+1)q^{k}-1+q^{k+1}}{q^{k}(2q-1)}$$

Again, we show that this does not hit any point in  $S_2$ . First it is greater than  $\frac{1}{q}$  since

$$\begin{array}{ll} q^{k-1}(2q-1) &< (l+1)q^k - 1 + q^{k+1} \\ 1 &< q^{k-1}\left((l+1)q + q^2 + (1-2q)\right) = q^{k-1}\left(q^2 + (l-1)q + 1\right), \end{array}$$

and less than 1 since

$$\begin{array}{rl} (l+1)q^k-1+q^{k+1} & < q^k(2q-1) \\ & -1 & < q^k\left(2q-1-(l+1)-q\right) = q^k\left(q-(l+2)\right). \end{array}$$

Take  $\gamma$  to be  $\varepsilon$  above  $bsc_k$  where  $0 < \varepsilon < \frac{q-1}{q^k(2q-1)}$ . Thus, we have connected the disconnected segments of  $f_r(cyl_k)$  to construct a waist curve of  $cyl_{k+1}$ .

Figure 8 shows the first few cylinders in this cylinder decomposition for  $r = \frac{1}{2}$ .

In the following theorem, we summarize the work done in this section by identifying our construction as a cylinder decomposition. In other words, we show that the closure of the cylinder decomposition is the entire surface. Recall that  $bsc_k$  is a concatenation of two saddle connections, for k > 1. Either  $cyl_k$  is adjacent to  $cyl_{k-1}$  or it is not. Theorem 3.2 (construction of  $bsc'_k$  for k > 1) tells us where  $cyl_k$  appears on  $\Box_1$  when it is not adjacent to  $cyl_{k-1}$ . In this case, we show that an infinite sequence of adjacent cylinders directly above this  $cyl_k$  converges to a part of some cylinder. Otherwise, the sequence of adjacent cylinders starting from  $cyl_1$  converges to part of the rigid spine.

**Theorem 4.6.** Given a geometric armadillo tail with parameter  $r = \frac{1}{q}$ , there exists a cylinder decomposition in the  $\frac{1}{2-r}$ -direction with a rigid spine.



FIGURE 8. Cylinder decomposition on the geometric armadillo tail (r = 1/2)

*Proof.* We first restrict to  $\Box_1$ , then use  $f_r$  to extend our results to the remaining squares.

For k > 1,  $bsc'_k$  hits the *y*-axis 2(q - 1) many times at points where the *y*-coordinate is  $\frac{i}{q^{k-1}(2q-1)}$ and  $\frac{i+q^k}{q^{k-1}(2q-1)}$ , for i = 1, ..., q - 1. (Theorem 3.2). This corresponds to the times  $cyl_k$  wraps around  $\Box_1$  when it is not adjacent to  $cyl_{k-1}$ . Since there is always a part of  $cyl_{k+1}$  that lies directly above  $cyl_k$ , we add the sum of skew widths of subsequent cylinders to observe the convergence to in  $\Box_1$ . We have

$$\frac{i}{q^{k-1}(2q-1)} + \frac{q-1}{q^k(2q-1)} + \frac{q-1}{q^{k+1}(2q-1)} + \dots = \frac{i+1}{q^{k-1}(2q-1)}.$$

The same holds for  $\frac{i+q^k}{q^{k-1}(2q-1)}$ . In other words, for  $i = 1, \ldots, q-2$ , the part of the cylinders in  $\Box_1$  converge to the bottom of a part of  $\operatorname{cyl}_k$ . Take i = q-1, then the same computation for  $\frac{q-1}{q^{k-1}(2q-1)}$  yields  $\frac{1}{q^{k-2}(2q-1)}$  which is a part of the bottom of  $\operatorname{cyl}_{k-1}$ , and for  $\frac{q-1+q^k}{q^{k-1}(2q-1)}$ , we get  $\frac{1+q^{k-1}}{q^{k-2}(2q-1)}$ , which is also a part of the bottom of  $\operatorname{cyl}_{k-1}$ . In other words, given  $\operatorname{cyl}_k \cap \Box_1$  not adjacent to  $\operatorname{cyl}_{k-1}$ , there is a sequence  $\operatorname{cyl}_i \cap \Box_1$  that converges to the bottom of  $\operatorname{cyl}_k \cap \Box_1$  or  $\operatorname{cyl}_{k-1} \cap \Box_1$ .

When k = 1, we refer to Theorem 2.3. The points at which  $bsc_1$  intersects the *y*-axis is at points where the *y*-coordinate is  $\frac{i}{2q-1}$  for i = 1, ..., 2q - 2 but not i = q. Then adding the sum of skewwidths

$$\frac{i}{2q-1} + \frac{q-1}{q(2q-1)} + \frac{q-1}{q^2(2q-1)} + \dots = \frac{i+1}{2q-1}$$

shows us that  $\text{cyl}_1 \cap \Box_1$  converges to either the bottom of  $\text{cyl}_1 \cap \Box_1$ , or a singularity (when i = 2q - 2), or the rigid spine (when i = q - 1).

Similar computations hold for  $\Box_2$ , and applying the  $f_r$  map implies that the same convergence applies in all squares.

# 5. MEASUREMENTS OF CYLINDERS (OR COMPUTATIONS REINFORCING WORK IN PREVIOUS SECTIONS)

In this section, we compute the length of the waist curve for each cylinder to find the modulus and area of each cylinder.

First, we will find the horizontal displacement of each waist curve. The table below lists the side lengths of each square and the number of times a waist curve of  $cyl_k$  goes through each square. This follows from the circle rotation defined in the previous section.

	$\square_1$	$\square_2$	 $\Box_i$	•••	$\Box_k$	$\square_{k+1}$
side length	1	1/q	$1/q^{i-1}$		$1/q^{k-1}$	$1/q^k$
#	k(2q-2) - 1	(k-1)(q-1)	(k-i+1)(q-1)		q-1	1

Then the horizontal displacement of a waist curve of  $\operatorname{cyl}_k$  is

$$\begin{split} & k(2q-2) - 1 + \frac{1}{q}(k-1)(q-1) + \frac{1}{q^2}(k-2)(q-1) + \dots + \frac{1}{q^{k-1}}(q-1) + \frac{1}{q^k} \\ &= k(2q-2) - 1 + \sum_{i=1}^{k-1} \frac{(k-i)(q-1)}{q^i} + \frac{1}{q^k} \\ &= k(2q-2) - 1 + \frac{q-1}{q^k} \sum_{i=1}^{k-1} (k-i)q^{k-i} + \frac{1}{q^k} \\ &= k(2q-2) - 1 + \frac{(k-1)q^{k+1} - kq^k + q}{q^k(q-1)} + \frac{1}{q^k}. \end{split}$$

For the last equality, we refer to the remark below.

Remark 5.1. The previous computation follows since:

$$\begin{split} \sum_{i=1}^{k-i} (k-i)q^{k-i} &= q+2q^2+3q^3+\dots+(k-1)q^{k-1} \\ &= q+2q^2+3q^3+\dots+(k-1)q^{k-1}+\left(q+\dots+q^{k-1}\right)-\left(q+\dots+q^{k-1}\right) \\ &= 2q+3q^2+\dots+kq^{k-1}-\frac{q(q^{k-1}-1)}{q-1} \\ &= \left(q^2+\dots+q^k\right)'-\frac{q^k-q}{q-1} \\ &= \left(\frac{q^2(q^{k-1}-1)}{q-1}\right)'-\frac{q^k-q}{q-1} \\ &= \frac{(k-1)q^{k+1}-kq^k+q}{(q-1)^2}. \end{split}$$

**Proposition 5.2.** The horizontal displacement of the waist curve of  $cyl_k$  is

$$(2q-1)\left(k-\frac{q^k-1}{q^k(q-1)}\right)$$

and the actual length of the waist curve, i.e., the circumference of  $cyl_k$  is

$$\left(k-\frac{q^k-1}{q^k(q-1)}\right)\sqrt{(2q-1)^2+q^2}.$$

Furthermore, the modulus of  $cyl_k$  is given as

$$\frac{\text{circumference}}{\text{width}} = \frac{q^2 + (2q-1)^2}{(q-1)^2} \left( kq^{k+1} - (k+1)q^k + 1 \right),$$

and the area of  $cyl_k$  is given by

area 
$$(\operatorname{cyl}_k) = \left(k - \frac{q^k - 1}{q^k(q - 1)}\right) \frac{q - 1}{q^k}.$$

**Proposition 5.3.** Given a geometric armadillo tail with parameter  $r = \frac{1}{q}$ ,  $q \in \mathbb{N} \setminus \{1\}$ , and a cylinder decomposition in the  $\frac{1}{2-r}$ -direction, the sum of skew-widths in  $\Box_1$  add up to 1.

*Proof.* The number of times  $cyl_k$  intersects the *y*-axis is 2kq - (2k + 1). Hence the sum of all skew-widths is:

$$\begin{split} \sum_{k=1}^{\infty} \left(2kq - (2k+1)\right) \cdot \frac{q-1}{q^k(2q-1)} &= \frac{q-1}{2q-1} \sum_{k=1}^{\infty} \left(\frac{2k}{q^{k-1}} - \frac{2k}{q^k} - \frac{1}{q^k}\right) \\ &= \frac{q-1}{2q-1} \left(\frac{2q^2}{(q-1)^2} - \frac{2q}{(q-1)^2} - \frac{1}{q-1}\right) = 1 \end{split}$$

for all  $q \in \mathbb{N} \setminus \{1\}$ . Moreover, as shown in Section 5, the area of the cylinders infinitely many cylinders matches the area of the surface. Consequently, we know that the closure of this set of infinitely many cylinders covers the surface, hence this is a cylinder decomposition.

Next, we verify that given r = 1/q, the infinite sum of area $(cyl_k)$  is equal to  $\frac{1}{1-r^2}$ , hence there exists an infinite cylinder decomposition in the  $\frac{1}{2-r}$ -direction.

**Proposition 5.4.** Given a geometric armadillo tail with parameter  $r = \frac{1}{q}$ ,  $q \in \mathbb{N} \setminus \{1\}$ , and a cylinder decomposition in the  $\frac{1}{2-r}$ -direction, the sum of areas add up to area of the surface.

*Proof.* From Proposition 5.2, we write area  $(cyl_k) = \frac{k(q-1)}{q^k} - \frac{1}{q^k} + \frac{1}{q^{2k}}$ . Following the same spirit as a previous remark, we use  $\sum_{i=1}^{\infty} ir^i = \frac{r}{(1-r)^2}$ , for |r| < 1. The sum of the first terms is

$$\sum_{k=1}^{\infty} \frac{k(q-1)}{q^k} = \frac{q}{q-1}$$

The second and third terms are geometric sequences, hence we have

$$\sum_{k=1}^{\infty} \operatorname{area}\left(\operatorname{cyl}_{k}\right) = \frac{q}{q-1} + \frac{1}{q-1} + \frac{1}{q^{2}-1} = \frac{q^{2}}{q^{2}-1}$$

our desired result.

## 6. NO PARABOLIC ELEMENT

Consider the horizontal cylinder decomposition of the armadillo tail seen in Figure 2. The *orthogonal cylinder decomposition* is comprised of exactly the squares. The element  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  is in the Veech group of the surface and this parabolic element corresponds to the orthogonal cylinder decomposition: indeed, the affine map associated with this Veech group element twists these cylinders, but preserves them as a set.

This phenomenon is understood in the finite translation surface setting, where the existence of a cylinder decomposition with rationally related moduli implies a parabolic element in the Veech group and vice-versa. Here, we see that in the orthogonal cylinder decomposition, the modulus of each cylinder is 1 since each cylinder is a square. However, the moduli of the cylinders in the horizontal cylinder decomposition in Figure 2 goes to infinity, and there is no parabolic element corresponding to that direction.



FIGURE 9. Cylinder decomposition C in the  $\frac{1}{2-r}$  direction

**Lemma 6.1.** Let C be a cylinder decomposition on a finite-area infinite translation surface. Then if the moduli of the cylinders tend to infinity, there is no parabolic element in the Veech group corresponding to an affine map that preserves the cylinder decomposition.

**Remark 6.2.** The lemma allows for rationally related moduli, hence distinguishes the finite translation surface setting from the infinite translation surface setting.

*Proof.* Assume otherwise. Then  $D\phi$  is a parabolic element in  $SL_2(\mathbb{R})$ , where the eigendirection corresponds to the direction of the cylinder decomposition. Up to conjugation,  $D\phi$  is of the form  $\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}$  for some p. However, there exists a cylinder in the decomposition with circumference greater than p, for any fixed p. This leads to a contradiction since the cylinder cannot be stabilized by  $D\phi$ .

**Corollary 6.3.** Let C be the cylinder decomposition constructed in the previous sections of this paper. There is no parabolic element in the Veech group corresponding to this cylinder decomposition.

*Proof.* We observe that the modulus of  $cyl_k$  goes to infinity as k goes to infinity.

**Remark 6.4.** In work of Hooper and Treviño [7], they observe that the golden ladder has a cylinder decomposition whose moduli are all equal, and the corresponding orthogonal cylinders are symmetric. They are able to find two parabolics, one in each direction. The construction of these parabolics was described by Thurston in the finite genus case. For the infinite genus case, see [7] or the Hooper–Thurston–Veech construction in [4].

**Lemma 6.5.** Let C be the cylinder decomposition constructed in the previous sections of this paper. There is no orthogonal cylinder decomposition.

*Proof.* The rigid spine runs along the top of what would be  $cyl_1^{\perp}$ . The length of the rigid spine is  $\frac{\sqrt{q^2+(2q-1)^2}}{q-1}$ . If q = 2, the spine is longer than the bottom saddle connection of the first cylinder, bsc<sub>1</sub>. Hence, there is no  $cyl_1^{\perp}$ . If q > 2, the spine is shorter than bsc<sub>1</sub>. If we pick a point in  $cyl_1$  and move vertically, we will pass through every cylinder and hit either the rigid spine or bsc<sub>1</sub> (see the proof of 4.6). The concatenation of the rigid spine and bsc<sub>1</sub> is clearly longer than bsc<sub>1</sub>. Consequently, there can be no  $cyl_1^{\perp}$ .

## Theorem 4.6 and Lemma 6.5 together prove the main theorem, Theorem 1.3.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74075 Email address: dami.lee@okstate.edu

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405 Email address: jwsouthe@iu.edu